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ON SOME FINITENESS QUESTIONS FOR ALGEBRAIC STACKS

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ABSTRACT. We prove that under a certain mild hypothesis, the DG category of D-modules on a quasi-compact algebraic stack is compactly generated. We also show that under the same hypothesis, the functor of global sections on the DG category of quasi-coherent sheaves is continuous.

Contents

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Introduction

0.1. **Introduction to the introduction.** This paper arose from an attempt to answer the following question: let $\mathcal{Y}$ be a quasi-compact algebraic stack over a field k of characteristic 0; is it true that the DG category of D-modules on $\mathcal{Y}$, denoted $\text{D-mod}(\mathcal{Y})$, is compactly generated?

We should remark that we did not pursue the above question out of pressing practical reasons: most (if not all) algebraic stacks that one encounters in practice are *perfect* in the sense of [BFN], and in this case the compact generation assertion is easy to prove and probably well-known. According to [BFN, Sect. 3.3], the class of perfect stacks is quite large. We decided to analyze the case of a general quasi-compact stack for aesthetic reasons.

0.1.1. Before we proceed any further let us explain why one should care about such questions as compact generation of a given DG category, and a description of its compact objects.

First, we should specify what is the world of DG categories that we work in. The world in question is that of DG categories that are cocomplete and continuous functors between them, see Sect. 0.6.2 for a brief review. The choice of this particular paradigm for DG categories appears to be a convenient framework in which to study various categorical aspects of algebraic geometry.

Compactness (resp., compact generation) are properties of an object in a given cocomplete DG category (resp., of a DG category). The relevance and usefulness of these notions in algebraic geometry was first brought to light in the paper of Thomason and Trobaugh, [TT].

The reasons for the importance of these notions can be summarized as follows: compact objects are those for which we can compute (or say something about) Hom out of them; and compactly generated categories are those for which we can compute (or say something about) continuous functors out of them.

0.1.2. The new results proved in the present paper fall into three distinct groups.

(i) Results about D-modules, that we originally started from, but which we treat last in the paper.

(ii) Results about the DG category of quasi-coherent sheaves on $\mathcal{Y}$, denoted $\text{QCoh}(\mathcal{Y})$, which are the most basic, and which are treated first.

(iii) Results about yet another category, namely, $\text{IndCoh}(\mathcal{Y})$, which forms a bridge between $\text{QCoh}(\mathcal{Y})$ and $\text{D-mod}(\mathcal{Y})$.

0.1.3. The logical structure of the paper is as follows:

Whatever we prove about $\text{QCoh}(\mathcal{Y})$ will easily imply the relevant results about $\text{IndCoh}(\mathcal{Y})$: for algebraic stacks the latter category differs only slightly from the former one.

The results about $\text{D-mod}(\mathcal{Y})$ are deduced from those about $\text{IndCoh}(\mathcal{Y})$ using a conservative forgetful functor $\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})} : \text{D-mod}(\mathcal{Y}) \to \text{IndCoh}(\mathcal{Y})$, which admits a left adjoint.

0.1.4. There is essentially only one piece of technology used in the proofs of all the main results: we stratify a given algebraic stack $\mathcal{Y}$ by locally closed substacks, which are essentially of the form Z/G, where Z is a quasi-compact scheme and G an algebraic group acting on it.

0.1.5. Finally, we should comment on why this paper came out so long (the first draft that contained all the main theorems had only five pages).

The reader will notice that the parts of the paper that contain any innovation (Sects. 2, 8 and 10) take less than one fifth of the volume.

The rest of the paper is either abstract nonsense (e.g., Sects. 4 and 9), or background material.

Some of the latter (e.g., the theory of D-modules on stacks) is included because we could not find adequate references in the literature. Some other things, especially various notions related to derived algebraic geometry, have been written down thanks to the work of Lurie and Toën-Vezzosi, but we decided to review them due to the novelty of the subject, in order to facilitate the job of the reader.

0.2. **Results on** $\text{D-mod}(\mathcal{Y})$**.**

0.2.1. We have not been able to treat the question of compact generation of $\text{D-mod}(\mathcal{Y})$ for arbitrary algebraic stacks. But we have obtained the following partial result (see Theorems 8.1.1 and 11.2.10):

Theorem 0.2.2. *Let $\mathcal{Y}$ be an algebraic stack of finite type over k. Assume that the automorphism groups of geometric points of $\mathcal{Y}$ are affine. Then* $\text{D-mod}(\mathcal{Y})$ *is compactly generated.*

0.2.3. In addition to this theorem, and under the above assumptions on $\mathcal{Y}$ (we call algebraic stacks with this property "QCA"), we prove a result characterizing the subcategory $\text{D-mod}(\mathcal{Y})^c$ of compact objects in $\text{D-mod}(\mathcal{Y})$ inside the larger category $\text{D-mod}_{\text{coh}}(\mathcal{Y})$ of coherent objects. (We were inspired by the following well known result: for any noetherian scheme Y, a bounded coherent object of $\text{QCoh}(Y)$ is compact if and only if it has finite Tor-dimension.)

We characterize $\text{D-mod}(\mathcal{Y})^c$ by a condition that we call *safety*, see Proposition 9.2.3 and Theorem 10.2.9. We note that safety of an object can be checked strata-wise: if $i : \mathcal{X} \hookrightarrow \mathcal{Y}$ is a closed substack and $j : (\mathcal{Y} - \mathcal{X}) \hookrightarrow \mathcal{Y}$ the complementary open, then an object $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$ is safe if and only if $i^!(\mathcal{F})$ and $j^!(\mathcal{F})$ are (see Corollary 10.4.3). However, the subcategory of safe objects is not preserved by the truncation functors with respect to the canonical t-structure on $\text{D-mod}(\mathcal{Y})$.

Furthermore, we prove Corollary 10.2.6 that characterizes those stacks $\mathcal{Y}$ of finite type over k for which the functor of global De Rham cohomlogy $\Gamma_{\text{dR}}(\mathcal{Y}, -)$ is continuous (i.e., commutes with colimits): this happens if and only if the neutral connected component of the automorphism group of any geometric point of $\mathcal{Y}$ is unipotent. We call such stacks *safe*. For example, any Deligne-Mumford stack is safe.

0.2.4. Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism between QCA algebraic stacks. The functor of D-module direct image $\pi_{\text{dR},*} : \text{D-mod}(Z_1) \to \text{D-mod}(Z_2)$ is in general not continuous, and consequently, it fails to have the base change property or satisfy the projection formula. In Sect. 9.3 we introduce a new functor $\pi_\blacktriangle$ of *renormalized direct image*, which fixes the above drawbacks of $\pi_{\text{dR},*}$. There always is a natural transformation $\pi_\blacktriangle \to \pi_{\text{dR},*}$, which is an isomorphism on safe objects.

0.3. **Results on** $\text{QCoh}(\mathcal{Y})$**.**

Let Vect denote the DG category of complexes of vector spaces over k.

0.3.1. We deduce Theorem 0.2.2 from the following more basic result about $\mathrm{QCoh}(\mathcal{Y})$ (see Theorem 1.4.2):

Theorem 0.3.2. *Let k be a field of characteristic 0 and let $\mathcal{Y}$ be a QCA algebraic stack of finite type over k. Then the (always derived) functor of global sections*

$$\Gamma(\mathcal{Y}, -) : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}$$

commutes with colimits. In other words, the structure sheaf $\mathcal{O}_{\mathcal{Y}}$ is a compact object of $\mathrm{QCoh}(\mathcal{Y})$*.*

We also obtain a relative version of Theorem 0.3.2 for morphisms of algebraic stacks $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ (see Corollary 1.4.5). It gives a sufficient condition for the functor

$$\pi_* : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2)$$

to commute with colimits (and thus have a base change property and satisfy the projection formula).

0.3.3. The question of compact generation of $\mathrm{QCoh}(\mathcal{Y})$ is subtle. It is easy to see that $\mathrm{QCoh}(\mathcal{Y})^c$ is contained in the category $\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$ of perfect complexes, and if $\mathcal{Y}$ satisfies the assumptions of Theorem 0.3.2 then $\mathrm{QCoh}(\mathcal{Y})^c = \mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$ (see Corollary 1.4.3). But we do not know if under these assumptions $\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$ always generates $\mathrm{QCoh}(\mathcal{Y})$. Ben-Zvi, Francis, and Nadler showed in [BFN, Section 3] that this is true for most of the stacks that one encounters in practice (e.g., see Lemma 2.6.3 below).

However, we were able to establish a property of $\mathrm{QCoh}(\mathcal{Y})$, which is weaker than compact generation, but still implies many of the favorable properties enjoyed by compactly generated categories (see Theorem 4.3.1):

Theorem 0.3.4. *Let $\mathcal{Y}$ be QCA algebraic stack. Then the category* $\mathrm{QCoh}(\mathcal{Y})$ *is* dualizable.

We refer the reader to Sect. 4.1.1 for a review of the notion of dualizable DG category.

0.3.5. In addition, we show that for a QCA algebraic stack $\mathcal{Y}$ and for any (pre)stack $\mathcal{Y}'$, the natural functor

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}')$$

is an equivalence (Corollary 4.3.4).

0.3.6. We should mention that in reviewing the above results about $\mathrm{QCoh}(\mathcal{Y})$ we were tacitly assuming that we were dealing with *classical algebraic stacks*. However, in the main body of the paper, we work in the setting of derived algebraic geometry, and henceforth by a "(pre)stack" we shall understand what one might call a "DG (pre)stack".

In particular, some caution is needed when dealing with the notion of algebraic stack of finite type, and for boundedness condition of the structure sheaf. We refer the reader to the main body of the text for the precise formulations of the above results in the DG context.

0.4. **Ind-coherent sheaves.** In addition to the categories $\mathrm{QCoh}(\mathcal{Y})$ and $\text{D-mod}(\mathcal{Y})$, there is a third player in this paper, namely, the DG category of ind-coherent sheaves, denoted $\mathrm{IndCoh}(\mathcal{Y})$. We refer the reader to [IndCoh] where this category is introduced and its basic properties are discussed.

As was mentioned in *loc.cit.*, Sects. 0.1 and 0.2, the assignment $\mathcal{Y} \mapsto \mathrm{IndCoh}(\mathcal{Y})$ is a natural sheaf-theoretic context in its own right. In particular, the category $\mathrm{IndCoh}(\mathcal{Y})$ is indispensable to treat the spectral side of the Geometric Langlands correspondence, see [AG].

In this paper the category $\mathrm{IndCoh}(\mathcal{Y})$ is used to prove Theorem 0.3.4. More importantly, this category serves as an intermediary between D-modules and $\mathcal{O}$-modules on $\mathcal{Y}$. Below we explain more details on the latter role of $\mathrm{IndCoh}(\mathcal{Y})$.

0.4.1. For an arbitrary (pre)stack, there is a naturally defined conservative forgetful functor

$$\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})} : \text{D-mod}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}),$$

and this functor is compatible with morphisms of (pre)stacks $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ under !-pullback functors on both sides.

Now, for a large class of prestacks, including algebraic stacks, the functor $\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}$ admits a left adjoint, denoted $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$. This adjoint pair of functors plays an important role in this paper: we use them to deduce Theorem 0.2.2 from Theorem 0.3.2.

0.4.2. The category IndCoh may be viewed as an accounting device that encodes the convergence of certain spectral sequences (equivalently, the basic properties of IndCoh established in [IndCoh], ensure that certain colimits commute with certain limits).

In light of this, the reader who is unfamiliar or not interested in the category IndCoh, may bypass it and relate the categories $\mathrm{QCoh}(\mathcal{Y})$ and $\text{D-mod}(\mathcal{Y})$ directly by the pairs of adjoint functors[1] $({}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}, {}'\mathbf{ind}_{\text{D-mod}(\mathcal{Y})})$ or $(\mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathcal{Y})}, \mathbf{ind}^{\text{left}}_{\text{D-mod}(\mathcal{Y})})$ introduced in Sects. 5.1.13 and 5.1.16 (for DG schemes), and 6.1.6 and 6.3.17 (for algebraic stacks). The corresponding variant of the proof of Theorem 0.2.2 is given in Sect. 8.2.

0.4.3. However, without the category $\mathrm{IndCoh}(\mathcal{Y})$, the treatment of $\text{D-mod}(\mathcal{Y})$ suffers from a certain awkwardness. Let us list three reasons for this in the ascending order of importance:

(i) Let Z be a scheme. The realization of $\text{D-mod}(Z)$ as "right" D-modules has the advantage of being compatible with the t-structure, see [GR1, Sect. 4.3] for a detailed discussion. So, let us say we want to work with right D-modules. However, if instead of $\mathrm{IndCoh}(Z)$ and the forgetful functor $\mathbf{oblv}_{\text{D-mod}(Z)}$ we use $\mathrm{QCoh}(Z)$ and the corresponding naive forgetful functor ${}'\mathbf{oblv}_{\text{D-mod}(Z)}$, we would not be able to formulate the compatibility of this forgetful functor with pullbacks. The reason is that for a general morphism of schemes $f : Z_1 \to Z_2$, the functor $f^!$ is defined and is continuous as a functor $\mathrm{IndCoh}(Z_2) \to \mathrm{IndCoh}(Z_1)$ but not as a functor $\mathrm{QCoh}(Z_2) \to \mathrm{QCoh}(Z_1)$.

(ii) The "left" forgetful functor $\mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathcal{Y})}$ is defined for any pre-stack $\mathcal{Y}$. However, it does not admit a left adjoint in many situations in which $\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}$ does, e.g., for ind-schemes. On the other hand, the naive "right" forgetful functor ${}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}$ is not defined unless $\mathcal{Y}$ is an *algebraic* stack.

(iii) As is explained in Sect. 5.2.2, the natural formalism[2] for the assignment $Z \mapsto \text{D-mod}(Z)$ is that of a functor from the category whose objects are schemes, and morphisms are correspondences between schemes. Moreover, we want this functor to be endowed with a natural transformation to one involving $\mathcal{O}$-modules (in either QCoh or IndCoh incarnation). However, the construction of this formalism carried out in [GR2] using IndCoh would run into serious problems if one tries to work with QCoh instead. [3]

[1] The pair $({}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}, {}'\mathbf{ind}_{\text{D-mod}(\mathcal{Y})})$ is related to the realization of $\text{D-mod}(\mathcal{Y})$ as "right" D-modules. The other pair is related to the realization as "left" D-modules.

[2] This formalism incorporates the base change isomorphism relating !-pullbacks and $*$-pushforwards.

[3] A part of the construction is that the functor of pullback under a closed embedding should admit a left adjoint; for this it is essential that we use the !-pullback and IndCoh as our category of $\mathcal{O}$-modules.

So, the upshot is that without IndCoh, we cannot really construct a workable formalism of D-modules, that allows to take both direct and inverse images.

0.4.4. Our main result concerning the category IndCoh is the following one (see Theorem 3.3.5):

Theorem 0.4.5. *For a QCA algebraic stack $\mathcal{Y}$, the category* $\operatorname{IndCoh}(\mathcal{Y})$ *is compactly generated. The category of its compact objects identifies with* $\operatorname{Coh}(\mathcal{Y})$.

In the above theorem, $\operatorname{Coh}(\mathcal{Y})$ is the full subcategory of $\operatorname{QCoh}(\mathcal{Y})$ of *coherent sheaves*, i.e., of bounded complexes with coherent cohomology. We deduce Theorem 0.4.5 from Theorem 0.3.2.

As we mentioned in Sect. 0.3.3, for a general QCA stack $\mathcal{Y}$ the problem of compact generation of $\operatorname{QCoh}(\mathcal{Y})$ is still open.

0.5. **Contents of the paper.**

0.5.1. In Sect. 1 we formulate the main technical result of this paper, Theorem 1.4.2.

We first fix our conventions regarding algebraic stacks. In Sects. 1 through 10 we adopt a definition of algebraic stacks slightly more restrictive than that of [LM]. Namely, we require the diagonal morphism to be schematic rather than representable.

We introduce the notion of QCA algebraic stack and of QCA morphism between arbitrary (pre)stacks.

We recall the definition of the category of $\operatorname{QCoh}(\mathcal{Y})$ for prestacks and in particular algebraic stacks.

We formulate Theorem 1.4.2, which is a sharpened version of Theorem 0.3.2 mentioned above. In Theorem 1.4.2 we assert not only that the functor $\Gamma(\mathcal{Y},-)$ is continuous, but also that it is of bounded cohomological amplitude.

We also show how Theorem 1.4.2 implies its relative version for a QCA morphism between (pre)stacks.

0.5.2. In Sect. 2 we prove Theorem 1.4.2. The idea of the proof is very simple. First, we show that the boundedness of the cohomological dimension implies the continuity of the functor $\Gamma(\mathcal{Y},-)$.

We then establish the required boundedness by stratifying our algebraic stack by locally closed substacks that are gerbes over schemes. For algebraic stacks of the latter form, one deduces the theorem directly by reducing to the case of quotient stacks Z/G, where Z is a quasi-compact scheme and G is a reductive group.

The char. 0 assumption is essential since we are using the fact that the category of representations of a reductive group is semi-simple.

0.5.3. In Sect. 3 we study the behavior of the category $\operatorname{IndCoh}(\mathcal{Y})$ for QCA algebraic stacks.

We first recall the definition and basic properties of $\operatorname{IndCoh}(\mathcal{Y})$.

We deduce Theorem 0.4.5 from Theorem 0.3.2.

We also introduce and study the direct image functor $\pi_*^{\operatorname{IndCoh}}$ for a morphism π between QCA algebraic stacks.

0.5.4. In Sect. 4 we prove (and study the implications of) the *dualizability* property of the categories $\mathrm{IndCoh}(\mathcal{Y})$ and $\mathrm{QCoh}(\mathcal{Y})$ for a QCA algebraic stack $\mathcal{Y}$.

We first recall the notion of dualizable DG category, and then deduce the dualizability of $\mathrm{IndCoh}(\mathcal{Y})$ from the fact that it is compactly generated.

We deduce the dualizability of $\mathrm{QCoh}(\mathcal{Y})$ from the fact that it is a retract of $\mathrm{IndCoh}(\mathcal{Y})$.

We then proceed to discuss Serre duality, which we interpret as a datum of equivalence of between $\mathrm{IndCoh}(\mathcal{Y})$ and its dual.

0.5.5. In Sect. 5 we review the theory of D-modules on (DG) schemes.

All of this material is well-known at the level of underlying triangulated categories, but unfortunately there is still no reference in the literature where all the needed constructions are carried out at the DG level. This is particularly relevant with regard to base change isomorphisms, where it is not straightforward to even formulate what structure they encode at the level of ∞-categories.

We also discuss Verdier duality for D-modules, which we interpret as a datum of equivalence between the category $\text{D-mod}(Z)$ and its dual, and its relation to Serre duality for $\mathrm{IndCoh}(Z)$.

0.5.6. In Sect. 6 we review the theory of D-modules on prestacks and algebraic stacks. This theory is also "well-known modulo homotopy-theoretic issues".

Having an appropriate formalism for the assignment $Z \rightsquigarrow \text{D-mod}(Z)$ for schemes, one defines the category $\text{D-mod}(\mathcal{Y})$ for an arbitrary prestack $\mathcal{Y}$, along with the naturally defined functors. The theory becomes richer once we restrict our attention to algebraic stacks; for example, in this case the category $\text{D-mod}(\mathcal{Y})$ has a t-structure.

For algebraic stacks we construct and study the induction functor

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} : \mathrm{IndCoh}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Y}),$$

left adjoint to the forgetful functor $\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}$. Its existence and properties are crucial for the proof of compact generation of $\text{D-mod}(\mathcal{Y})$ on QCA algebraic stacks, as well as for the relation between the conditions of compactness and safety for objects of $\text{D-mod}(\mathcal{Y})$, and for the construction of the renormalized direct image functor. In short, the functor $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ produces a supply of objects of $\text{D-mod}(\mathcal{Y})$ whose cohomological behavior we can control.

0.5.7. In Sect. 7 we define the functor of de Rham cohomology $\Gamma_{\mathrm{dR}}(\mathcal{Y}, -) : \text{D-mod}(\mathcal{Y}) \to \mathrm{Vect}$, where $\mathcal{Y}$ is an algebraic stack, and discuss its failure to be continuous.

Furthermore, we generalize this to the case of the D-module direct image functor $\pi_{\mathrm{dR},*}$ for a morphism π between algebraic stacks.

We also discuss the condition of *coherence* on an object of $\text{D-mod}(\mathcal{Y})$, and we explain that for quasi-compact algebraic stacks, unlike quasi-compact schemes, the inclusion

$$\text{D-mod}(\mathcal{Y})^c \subset \text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$$

is *not* an equality.

0.5.8. In Sect. 8 we prove Theorem 0.2.2. More precisely, we show that for a QCA algebraic stack $\mathcal{Y}$, the category $\text{D-mod}(\mathcal{Y})$ is compactly generated by objects of the form $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})$ for $\mathcal{F} \in \text{Coh}(\mathcal{Y})$.

We also show that Theorem 0.2.2, combined with a compatibility of Serre and Verdier dualities, imply that for a QCA algebraic stack $\mathcal{Y}$, the category $\text{D-mod}(\mathcal{Y})$ is equivalent to its dual, as was the case for schemes.

Finally, we show that for $\mathcal{Y}$ as above and any prestack $\mathcal{Y}'$, the natural functor

$$\text{D-mod}(\mathcal{Y}) \times \text{D-mod}(\mathcal{Y}') \to \text{D-mod}(\mathcal{Y} \times \mathcal{Y}')$$

is an equivalence.

0.5.9. In Sect. 9 we introduce the functors of renormalized de Rham cohomology and, more generally, renormalized D-module direct image for morphisms between QCA algebraic stacks.

We show that both these functors can be defined as ind-extensions of restrictions of the original functors $\Gamma_{\text{dR}}(\mathcal{Y}, -)$ and $\pi_{\text{dR},*}$ to the subcategory of compact objects.

We show that the renormalized direct image functor $\pi_\blacktriangle$, unlike the original functor $\pi_{\text{dR},*}$, has the base change property and satisfies the projection formula.

We introduce the notion of *safe* object of $\text{D-mod}(\mathcal{Y})$, and we show that for safe objects $\pi_\blacktriangle(\mathcal{M}) \simeq \pi_{\text{dR},*}(\mathcal{M})$.

We also show that compact objects of $\text{D-mod}(\mathcal{Y})$ can be characterized as those objects of $\text{D-mod}_{\text{coh}}(\mathcal{Y})$ that are also safe.

Finally, we show that the functor $\pi_\blacktriangle$ exhibits a behavior opposite to that of $\pi_{\text{dR},*}$ with respect to its cohomological amplitude: the functor $\pi_{\text{dR},*}$ is left t-exact, up to a cohomological shift, whereas the functor $\pi_\blacktriangle$ is right t-exact, up to a cohomological shift.

0.5.10. In Sect. 10 we give geometric descriptions of safe algebraic stacks (i.e., those QCA stacks, for which all objects of $\text{D-mod}(\mathcal{Y})$ are safe), and a geometric criterion for safety of objects of $\text{D-mod}(\mathcal{Y})$ in general. The latter description also provides a more explicit description of compact objects of $\text{D-mod}(\mathcal{Y})$ inside $\text{D-mod}_{\text{coh}}(\mathcal{Y})$.

We prove that a quasi-compact algebraic stack $\mathcal{Y}$ is safe if and only if the neutral components of stabilizers of its geometric points are unipotent. In particular, any Deligne-Mumford quasi-compact algebraic stack is safe.

The criterion for safety of an object, roughly, looks as follows: a cohomologically bounded object $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$ is safe if and only if for every point $y \in \mathcal{Y}$ with $G_y = \text{Aut}(y)$, the restriction $\mathcal{M}|_{BG_y}$ (here BG_y denotes the classifying stack of G_y which maps canonically into $\mathcal{Y}$) has the property that

$$\pi_{\text{dR},*}(\mathcal{M}|_{BG_y})$$

is still cohomologically bounded, where π denotes the map $BG_y \to B\Gamma_y$, where $\Gamma_y = \pi_0(G_y)$.

Conversely, we show that every cohomologically bounded safe object of $\text{D-mod}(\mathcal{Y})$ can be obtained by a finite iteration of taking cones starting from objects of the form $\phi_{\text{dR},*}(\mathcal{N})$, where $\phi : S \to \mathcal{Y}$ with S being a quasi-compact scheme and $\mathcal{N} \in \text{D-mod}(S)^b$.

0.5.11. Finally, in Sect. 11 we explain how to generalize the results of Sects. 1-10 to the case of algebraic stacks in the sense of [LM]; we call the latter LM-algebraic stacks.

Namely, we explain that since quasi-compact algebraic spaces are QCA when viewed as algebraic stacks, they can be used as building blocks for the categories QCoh(−), IndCoh(−) and D-mod(−) instead of schemes. This will imply that the proofs of all the results of this paper are valid for QCA LM-algebraic stacks and morphisms.

0.6. **Conventions, notation and terminology.** We will be working over a fixed ground field k of characteristic 0. Without loss of generality one can assume that k is algebraically closed.

0.6.1. ∞-*categories.* Throughout the paper we shall be working with $(\infty,1)$-categories. Our treatment is not tied to any specific model, but we shall use [Lu1] as our basic reference.

We let ∞-Grpd denote the ∞-category of ∞-groupids, a.k.a. "spaces".

If $\mathbf{C}$ is an ∞-category and $\mathbf{c}_1,\mathbf{c}_2\in\mathbf{C}$ are objects, we shall denote by $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)$ the ∞-groupoid of maps between these two objects. We shall use the notation $\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)\in\mathrm{Sets}$ for $\pi_0(\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2))$, i.e., Hom in the homotopy category.

We shall often say "category" when in fact we mean an ∞-category.

If $F:\mathbf{C}'\to\mathbf{C}$ is a functor between ∞-categories, we shall say that F is *0-fully faithful* (or just *fully faithful*) if F induces an equivalence on $\mathrm{Maps}(-,-)$. In this case we call the essential image of $\mathbf{C}'$ a *full subcategory* of $\mathbf{C}$.

We shall say that F is *1-fully faithful* (or just *faithful*) if F induces a *monomorphism* on $\mathrm{Maps}(-,-)$, i.e., if the map

$$\mathrm{Maps}_{\mathbf{C}'}(\mathbf{c}'_1,\mathbf{c}'_2)\to\mathrm{Maps}_{\mathbf{C}}(F(\mathbf{c}'_1),F(\mathbf{c}'_2)) \tag{0.1}$$

is the inclusion of a union of some of the connected components. If, moreover, the map (0.1) is surjective on those connected components of $\mathrm{Maps}_{\mathbf{C}}(F(\mathbf{c}'_1),F(\mathbf{c}'_2))$ that correspond to isomorphisms, we shall refer to the essential image of $\mathbf{C}'$ as a *1-full subcategory* of $\mathbf{C}$.

0.6.2. *DG categories: elementary aspects.* We will be working with DG categories over k. Unless explicitly specified otherwise, all DG categories will be assumed cocomplete, i.e., contain infinite direct sums (equivalently, filtered colimits, and equivalently all colimits).

We let Vect denote the DG category of complexes of k-vector spaces.

For a DG category $\mathbf{C}$, and $\mathbf{c}_1,\mathbf{c}_2\in\mathbf{C}$ we can form the object $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)\in\mathrm{Vect}$. We have

$$\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)\simeq\tau^{\leq 0}\left(\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)\right),$$

where in the right-hand side we regard an object of $\mathrm{Vect}^{\leq 0}$ as an object of ∞-Grpd via the Dold-Kan functor.

We shall use the notation $\mathrm{Hom}^{\bullet}_{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)$ to denote the graded vector space

$$\underset{i}{\oplus}H^i\left(\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2)\right)\simeq\underset{i}{\oplus}\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}_1,\mathbf{c}_2[i]).$$

We shall often use the notion of t-structure on a DG category. For $\mathbf{C}$ endowed with a t-structure, we shall denote by $\mathbf{C}^{\leq 0}$, $\mathbf{C}^{\geq 0}$, $\mathbf{C}^{-}$, $\mathbf{C}^{+}$, $\mathbf{C}^{b}$ the corresponding subcategories of connective, coconnective, eventually connective (a.k.a. bounded above), eventually coconnective (a.k.a. bounded below) and cohomologically bounded objects.

We let $\mathbf{C}^{\heartsuit}$ denote the abelian category equal to the heart (a.k.a. core) of the t-structure. For example, $\mathrm{Vect}^{\heartsuit}$ is the usual category of k-vector spaces.

0.6.3. *Functors.* All functors between DG categories considered in this paper, without exception, will be exact (i.e., map exact triangles to exact triangles). [4]

It is a corollary of the adjoint functor theorem a cocomplete DG category also contains all *limits*, see [Lu1, Corollary 5.5.2.4].

More generally, we have a version of Brown's representability theorem that says that any exact contravariant functor $F : \mathbf{C} \to \mathrm{Vect}$ is ind-representable (see [Lu1, Corollary 5.3.5.4]), and it is representable if and only if F takes colimits in $\mathbf{C}$ to limits in Vect.

0.6.4. *Continuous functors.* For two DG categories $\mathbf{C}_1$, $\mathbf{C}_2$ we shall denote by $\mathrm{Funct}(\mathbf{C}_1, \mathbf{C}_2)$ the DG category of all (exact) functors $\mathbf{C}_1 \to \mathbf{C}_2$, and by $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_1, \mathbf{C}_2)$ its full DG subcategory consisting of *continuous* functors, i.e., those functors that commute with infinite direct sums (equivalently, filtered colimits, and equivalently all colimits).

By default, whenever we talk about a functor between DG categories, we will mean a continuous functor. We shall also encounter non-continuous functors, but we will explicitly emphasize whenever this happens.

The importance of continuous functors vs. all functors is, among the rest, in the fact that the operation of tensor product of DG categories, reviewed in Sect. 4.1.1, is functorial with respect to continuous functors.

0.6.5. *Compactness.* We recall that an object $\mathbf{c}$ in a DG category is called *compact* if the functor

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \mathrm{Vect}^{\heartsuit}$$

commutes with direct sums. This is equivalent to requiring that the functor

$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \mathrm{Vect}$$

be continuous, and still equivalent to requiring that the functor $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \infty\text{-Grpd}$ commute with filtered colimits; the latter interpretation of compactness makes sense for an arbitary ∞-category closed under filtered colimits. We let $\mathbf{C}^c$ denote the full *but not cocomplete* subcategory of $\mathbf{C}$ spanned by compact objects. [5]

A DG category $\mathbf{C}$ is said to be compactly generated if there exists a set of compact objects $\mathbf{c}_\alpha \in \mathbf{C}$ that generate it, i.e.,

$$\mathcal{M}aps(\mathbf{c}_\alpha, \mathbf{c}) = 0 \Rightarrow \mathbf{c} = 0.$$

Equivalently, if $\mathbf{C}$ does not contain proper full cocomplete subcategories that contain all the objects $\mathbf{c}_\alpha$.

0.6.6. *DG categories: homotopy-theoretic aspects.* We shall regard the totality of DG categories as an $(\infty, 1)$-category in two ways, denoted DGCat and $\mathrm{DGCat}_{\mathrm{cont}}$. In both cases the objects are DG categories. In the former case, we take as 1-morphisms all (exact) functors, whereas in the latter case we take those (exact) functors that are continuous. The latter is a 1-full subcategory of the former.

The above framework for the theory of DG categories is not fully documented (see, however, [GL:DG] where the basic facts are summarized). For a better documented theory, one can replace the ∞-category of DG categories by that of stable ∞-categories tensored over k (the latter theory is defined as a consequence of [Lu2, Sects. 4.2 and 6.3]).

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[4] As a way to deal with set-theoretic issues, we will assume that all our DG categories are presentable, and all functors between them are accessible (see [Lu1, Definitions 5.4.2.5 and 5.5.0.1]); an assumption which is always satisfied in practice.

[5] The presentability assumption on $\mathbf{C}$ implies that $\mathbf{C}^c$ is small.

0.6.7. *Ind-completions.* If $\mathbf{C}^0$ is a small, and hence, *non-cocomplete*, DG category, one can canonically attach to it a cocomplete one, referred to as the *ind-completion* of $\mathbf{C}^0$, denoted $\mathrm{Ind}(\mathbf{C}^0)$, and characterized by the property that for $\mathbf{C} \in \mathrm{DGCat}_{\mathrm{cont}}$

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Ind}(\mathbf{C}^0), \mathbf{C})$$

is the category of *all* (exact) functors $\mathbf{C}^0 \to \mathbf{C}$. For a functor $F : \mathbf{C}^0 \to \mathbf{C}$, the resulting continuous functor $\mathrm{Ind}(\mathbf{C}^0) \to \mathbf{C}$ is called the "ind-extension of F".

The objects of $\mathbf{C}^0$ are compact when viewed as objects of $\mathbf{C}$. It is not true, however, that the inclusion $\mathbf{C}^0 \subset \mathbf{C}^c$ is equality. Rather, $\mathbf{C}^c$ is the Karoubian completion of $\mathbf{C}^0$, i.e., every object of the former can be realized as a direct summand of an object of the latter (see [N, Theorem 2.1] or [BeV, Prop. 1.4.2] for the proof).

A DG category is compactly generated if and only if it is of the form $\mathrm{Ind}(\mathbf{C}^0)$ for $\mathbf{C}^0$ as above.

0.6.8. *DG Schemes.* Throughout the paper we shall work in the context of derived algebraic geometry over the field k. We denote $\mathrm{Spec}(k) =: \mathrm{pt}$.

We shall denote by DGSch, $\mathrm{DGSch}_{\text{qs-qc}}$ and $\mathrm{DGSch}^{\mathrm{aff}}$ the categories of DG schemes, quasi-separated and quasi-compact DG schemes, and affine DG schemes, respectively. The fundamental treatment of these objects can be found in [Lu3]. For a brief review see also [GL:Stacks], Sect. 3. The above categories contain the full subcategories Sch, $\mathrm{Sch}_{\text{qs-qc}}$ and $\mathrm{Sch}^{\mathrm{aff}}$ of classical schemes.

For the reader's convenience, let us recall the notions of smoothness and flatness in the DG setting.

A map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ between affine DG schemes is said to be flat if $H^0(B)$ is flat as a module over $H^0(A)$, plus the following equivalent conditions hold:

- The natural map $H^0(B) \underset{H^0(A)}{\otimes} H^i(A) \to H^i(B)$ is an isomorphism for every i.
- For any A-module M, the natural map $H^0(B) \underset{H^0(A)}{\otimes} H^i(M) \to H^i(B \underset{A}{\otimes} M)$ is an isomorphism for every i.
- If an A-module N is concentrated in degree 0 then so is $B \underset{A}{\otimes} N$.

The above notion is easily seen to be local in the Zariski topology in both $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$. The notion of flatness for a morphism between DG schemes is defined accordingly.

Let $f : S_1 \to S_2$ be a morphism of DG schemes. We shall say that it is smooth/flat almost of finite presentation if the following conditions hold:

- f is flat (in particular, the base-changed DG scheme ${}^{cl}S_2 \underset{S_2}{\times} S_1$ is classical), and
- the map of classical schemes ${}^{cl}S_2 \underset{S_2}{\times} S_1 \to {}^{cl}S_2$ is smooth/flat of finite presentation.

In the above formulas, for a DG scheme S, we denote by ${}^{cl}S$ the underlying classical scheme. I.e., locally, if $S = \mathrm{Spec}(A)$, then ${}^{cl}S = \mathrm{Spec}(H^0(A))$.

A morphism $f : S_1 \to S_2$ is said to be fppf if it is flat almost of finite presentation and surjective at the level of the underlying classical schemes.

0.6.9. *Stacks and prestacks.* By a prestack we shall mean an arbitrary functor

$$\mathcal{Y} : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}\,.$$

We denote the category of prestacks by PreStk.

We should emphasize that the reader who is reluctant do deal with functors taking values in ∞-groupoids, and who is willing to pay the price of staying within the world of classical algebraic geometry, may ignore any mention of prestacks, and replace them by functors with values in usual (i.e., 1-truncated) groupoids.

A prestack is called a stack if it satisfies fppf descent, see [GL:Stacks], Sect. 2.2. We denote the full subcategory of PreStk formed by stacks by Stk. The embedding $\mathrm{Stk} \hookrightarrow \mathrm{PreStk}$ admits a left adjoint, denoted L, and called a sheafification functor.

That said, the distinction between stacks and prestacks will not play a significant role in this paper, because for a prestack $\mathcal{Y}$, the canonical map $\mathcal{Y} \to L(\mathcal{Y})$ induces an equivalence on the category $\mathrm{QCoh}(-)$. The same happens for $\mathrm{IndCoh}(-)$ and $\mathrm{D}\text{-mod}(-)$ in the context of prestacks locally almost of finite type, considered starting from Sects. 3-10 on.

We can also consider the category of classical prestacks, denoted ${}^{cl}\mathrm{PreStk}$, the latter being the category of all functors

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}\,.$$

We have a natural restriction functor

$$\mathrm{Res}_{\mathrm{cl}\to\mathrm{DG}} : \mathrm{PreStk} \to {}^{cl}\mathrm{PreStk},$$

which admits a fully faithful left adjoint, given by the procedure of *left Kan extension*, see [GL:Stacks], Sect. 1.1.3. Let us denote this functor $\mathrm{LKE}_{\mathrm{cl}\to\mathrm{DG}}$. Thus, the functor $\mathrm{LKE}_{\mathrm{cl}\to\mathrm{DG}}$ allows us to view ${}^{cl}\mathrm{PreStk}$ as a full subcategory of PreStk.

For example, the composition of the Yoneda embedding $\mathrm{Sch}^{\mathrm{aff}} \to {}^{cl}\mathrm{PreStk}$ with $\mathrm{LKE}_{\mathrm{cl}\to\mathrm{DG}}$ is the composition of the tautological embedding $\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{DGSch}^{\mathrm{aff}}$, followed by the Yoneda embedding $\mathrm{DGSch}^{\mathrm{aff}} \to \mathrm{PreStk}$.

We also have the corresponding full subcategory ${}^{cl}\mathrm{Stk} \subset {}^{cl}\mathrm{PreStk}$. The functor $\mathrm{Res}_{\mathrm{cl}\to\mathrm{DG}}$ sends $\mathrm{Stk} \subset \mathrm{PreStk}$ to ${}^{cl}\mathrm{Stk} \subset {}^{cl}\mathrm{PreStk}$. However, the functor $\mathrm{LKE}_{\mathrm{cl}\to\mathrm{DG}}$ does *not* necessarily send ${}^{cl}\mathrm{Stk}$ to Stk.

Following [GL:Stacks], Sect. 2.4.7, we shall call a stack *classical* if it can be obtained as a sheafification of a classical prestack. This is equivalent to the condition that the natural map

$$L(\mathrm{LKE}_{\mathrm{cl}\to\mathrm{DG}} \circ \mathrm{Res}_{\mathrm{cl}\to\mathrm{DG}}(\mathcal{Y})) \to \mathcal{Y}$$

be an isomorphism.

In particular, it is not true that a classical non-affine DG scheme is classical as a prestack. But it is classical as a stack.

When in the main body of the text we will talk about algebraic stacks, the condition of being classical is understood in the above sense.

For a p=prestack/stack/DG scheme/affine DG scheme $\mathcal{Y}$, the expression "the classical p underlying $\mathcal{Y}$" means the object $\mathrm{Res}_{\mathrm{cl}\to\mathrm{DG}}(\mathcal{Y}) \in {}^{cl}\mathrm{PreStk}$ that belongs to the appropriate full subcategory

$$\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{Sch} \subset \mathrm{Stk} \subset \mathrm{PreStk}\,.$$

We will use a shorthand notation for this operation: $\mathcal{Y} \mapsto {}^{cl}\mathcal{Y}$.

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1. Results on $\mathrm{QCoh}(\mathcal{Y})$

In Sects. 1.1-1.3 we introduce the basic definitions and recall some well-known facts. The new results are formulated in Sect. 1.4.

1.1. **Assumptions on stacks.**

1.1.1. *Algebraic stacks.* In Sections 1-10 we will use the following definition of algebraicity of a stack, which is slightly more restrictive than that of [LM] (in the context of classical stacks) or [GL:Stacks, Sect. 4.2.8] (in the DG context).

1.1.2. First, recall that a morphism $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ between prestacks is called schematic if for any affine DG scheme S equipped with a morphism $S \to \mathcal{Y}_2$ the prestack $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$ is a DG scheme. The notions of surjectivity/flatness/smoothness/quasi-compactness/quasi-separatedness make sense for schematic morphisms: π has one of the above properties if for every $S \to \mathcal{Y}_2$ as above, the map of DG schemes $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S$ has the corresponding property.

1.1.3. Let $\mathcal{Y}$ be a stack. We shall say that $\mathcal{Y}$ is a algebraic if

- The diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is schematic, quasi-separated and quasi-compact.
- There exists a DG scheme Z and a map $f : Z \to \mathcal{Y}$ (automatically schematic, by the previous condition) such that f is smooth and surjective.

A pair (Z, f) as above is called a *presentation* or *atlas* for $\mathcal{Y}$.

Remark 1.1.4. In [LM] one imposes a slightly stronger condition on the diagonal map $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$. Namely, in *loc.cit.* it is required to be separated rather than quasi-separated. However, the above weaker condition seems more natural, and it will suffice for our purposes (the latter being Lemma 2.5.2, that relies on [LM, Corollary 10.8], while the latter does not require the separated diagonal assumption).

Remark 1.1.5. To get the more general notion of algebraic stack in the spirit of [LM] (for brevity, *LM-algebraic* stack), one replaces the word "schematic" in the above definition by "representable", see Sect. 11.1.3. [6] In fact, *all the results formulated in this paper are valid for LM-algebraic stacks;* in Sect. 11 we shall explain the necessary modifications. On the other hand, most LM-algebraic stacks one encounters in practice satisfy the more restrictive definition as well. The advantage of LM-algebraic stacks vs. algebraic stacks defined above is that the former, unlike the latter, satisfy fppf descent.

To recover the even more general notion of algebraic stack (a.k.a. 1-Artin stack) from [GL:Stacks, Sect. 4.2.8], one should omit the condition on the diagonal map to be quasi-separated and quasi-compact. However, these conditions are essential for the validity of the results in this paper.

[6] A morphism $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ between prestacks is called representable if for every affine DG scheme S equipped with a morphism $S \to \mathcal{Y}_2$ the prestack $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$ is an algebraic space, see Sect. 11.1.1 for a review of the latter notion in the context of derived algebraic geometry.

Definition 1.1.6. *We shall say that an algebraic stack $\mathcal{Y}$ is quasi-compact if admits an atlas (Z, f), where Z is an affine (equivalently, quasi-compact) DG scheme.*

1.1.7. *QCA stacks.*

QCA is shorthand for "quasi-compact and with affine automorphism groups".

Definition 1.1.8. *We shall say that algebraic stack $\mathcal{Y}$ is QCA if*

1. *It is quasi-compact;*
2. *The automorphism groups of its geometric points are affine;*
3. *The classical inertia stack, i.e., the classical algebraic stack ${}^{cl}(\mathcal{Y} \underset{\mathcal{Y}\times\mathcal{Y}}{\times} \mathcal{Y})$, is of finite presentation over ${}^{cl}\mathcal{Y}$.*

In particular, any algebraic space automatically satisfies this condition (indeed, the classical inertia stack of an algebraic space $\mathcal{X}$ is isomorphic to ${}^{cl}\mathcal{X}$). In addition, it is clear that if

$$
\begin{array}{ccc}
\mathcal{Y}' & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{X}' & \longrightarrow & \mathcal{X},
\end{array}
$$

is a Cartesian diagram, where $\mathcal{X}$ and $\mathcal{X}'$ are algebraic spaces, and $\mathcal{Y}$ is a QCA algebraic stack, then so is $\mathcal{Y}'$.

The class of QCA algebraic stacks will play a fundamental role in this article. We also need the relative version of the QCA condition.

Definition 1.1.9. *We shall say that a morphism $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ between prestacks is QCA if for every affine DG scheme S and a morphism $S \to \mathcal{Y}_2$, the base-changed prestack $\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} S$ is an algebraic stack and is QCA.*

For example, it is easy to show that if $\mathcal{Y}_1$ is a QCA algebraic stack and $\mathcal{Y}_2$ is any algebraic stack, then any morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ is QCA.

1.2. **Quasi-coherent sheaves.**

1.2.1. *Definition.* Let $\mathcal{Y}$ be any prestack. Let us recall (see e.g. [GL:QCoh, Sect. 1.1.3]) that the category $\mathrm{QCoh}(\mathcal{Y})$ is defined as

$$\underset{(S,g)\in((\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \mathrm{QCoh}(S). \tag{1.1}$$

Here $(\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$ is the category of pairs (S, g), where S is an affine DG scheme, and g is a map $S \to \mathcal{Y}$.

1.2.2. Let us comment on the structure of the above definition:

We view the assignment $(S, g) \rightsquigarrow \mathrm{QCoh}(S)$ as a functor between ∞-categories

$$((\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}, \tag{1.2}$$

and the limit is taken in the $(\infty, 1)$-category $\mathrm{DGCat}_{\mathrm{cont}}$. The functor (1.2) is obtained by restriction under the forgetful map $(\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}} \to \mathrm{DGSch}^{\mathrm{aff}}$ of the functor

$$\mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}} : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

where for $f : S' \to S$, the map $\mathrm{QCoh}(S) \to \mathrm{QCoh}(S')$ is f^*. (The latter functor can be constructed in a "hands-on" way; this has been carried out in detail in [Lu3].)

In other words, an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ is an assignment for any $(S, g : S \to \mathcal{Y})$ of an object $\mathcal{F}|_S := g^*(\mathcal{F}) \in \mathrm{QCoh}(S)$, and a homotopy-coherent system of isomorphisms

$$f^*(g^*(\mathcal{F})) \simeq (g \circ f)^*(\mathcal{F}) \in \mathrm{QCoh}(S'),$$

for maps of DG schemes $f : S' \to S$.

Remark 1.2.3. For $\mathcal{Y}$ classical and algebraic, the definition of $\mathrm{QCoh}(\mathcal{Y})$ given above is different from the one of [LM] (in *loc.cit.*, at the level of triangulated categories, $\mathrm{QCoh}(\mathcal{Y})$ is defined as a full subcategory in the derived category of the abelian category of sheaves of $\mathcal{O}$-modules on the smooth site of $\mathcal{Y}$). It is easy to show that the eventually coconnective (=bounded from below) parts of both categories, i.e., the two versions of $\mathrm{QCoh}(\mathcal{Y})^+$, are canonically equivalent. However, we have no reasons to believe that the entire categories are equivalent in general. The reason that we insist on considering the entire category $\mathrm{QCoh}(\mathcal{Y})$ is that this paper is largely devoted to the notion of compactness, which only makes sense in a cocomplete category.

Remark 1.2.4. In the definition of $\mathrm{QCoh}(\mathcal{Y})$, one can replace the category $\mathrm{DGSch}^{\mathrm{aff}}$ of affine DG schemes by either $\mathrm{DGSch}_{\text{qs-qc}}$ or of quasi-separated and quasi-compact DG schemes or just DGSch of all DG schemes. The limit category will not change due to the Zariski descent property of the assignment $S \rightsquigarrow \mathrm{QCoh}(S)$.

1.2.5. In the definition of $\mathrm{QCoh}(\mathcal{Y})$ it is often convenient to replace the category $\mathrm{DGSch}_{/\mathcal{Y}}$ (resp., $(\mathrm{DGSch}_{\text{qs-qc}})_{/\mathcal{Y}}$, $(\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$) by a another category A, equipped with a functor

$$(a \in A) \mapsto (S_a, g_a)$$

to $\mathrm{DGSch}_{/\mathcal{Y}}$ (resp., $(\mathrm{DGSch}_{\text{qs-qc}})_{/\mathcal{Y}}$, $(\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$), (provided that the limit will be the same). Below are several examples that will be used in this paper.

(i) If $\mathcal{Y}$ is classical (as a stack or a prestack), one can take the category $A := (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$, equipped with the tautological inclusion to $(\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$. I.e., we replace DG schemes by classical schemes. Indeed, if $\mathcal{Y}$ is classical as a prestack, the fact that the limit category will be the same follows the from fact that the property of $\mathcal{Y}$ to be classical means that the inclusion $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}} \to (\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$ is cofinal (i.e., for every $S \in \mathrm{DGSch}^{\mathrm{aff}}$ and a point $y : S \to \mathcal{Y}$, there exists a factorization $S \to S' \to \mathcal{Y}$, where $S' \in \mathrm{Sch}^{\mathrm{aff}}$, and the category of such factorizations is contractible.) For stacks, this follows from the fact that the map $\mathcal{Y} \to L(\mathcal{Y})$, where we remind that $L(-)$ denotes fppf sheafification, induces an isomorphism on QCoh.

(ii) Let $\mathcal{Y} \to \mathcal{Y}'$ be a schematic (resp., schematic + quasi-separated and quasi-compact; affine) map between prestacks.

Then we can take A to be $\mathrm{DGSch}_{/\mathcal{Y}'}$ (resp., $(\mathrm{DGSch}_{\text{qs-qc}})_{/\mathcal{Y}'}$; $(\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}'}$) via the functor

$$S' \mapsto S := S' \underset{\mathcal{Y}'}{\times} \mathcal{Y}. \tag{1.3}$$

Indeed, it is easy to see that the above functor is cofinal.

(iii) Suppose that $\mathcal{Y}$ is algebraic and let $f : Z \to \mathcal{Y}$ be an fppf atlas. Then we can replace $\mathrm{DGSch}_{/\mathcal{Y}}$ by the Čech nerve of f. The fact that the limit category is the same follows from the fppf descent for QCoh on DG schemes.

(iv) Assume again that $\mathcal{Y}$ is algebraic. We can take A to be the 1-full subcategory

$$\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}} \subset \mathrm{DGSch}_{/\mathcal{Y}},$$

or, respectively,

$$(\mathrm{DGSch}_{\text{qs-qc}})_{/\mathcal{Y},\mathrm{smooth}} \subset (\mathrm{DGSch}_{\text{qs-qc}})_{/\mathcal{Y}},\ (\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y},\mathrm{smooth}} \subset (\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}},$$

where we restrict objects to those (S,g), for which g is smooth, and 1-morphisms to those $f: S_1 \to S_2$, for which f is smooth. The fact that limit category is the same is shown in [IndCoh, Sect. 11.2 and particularly Corollary 11.2.3]. The word "smooth" can also be replaced by the word "flat". The same proof applies to establish the following generalization:

Lemma 1.2.6. *Let $\mathcal{Y} \to \mathcal{Y}'$ be a schematic (resp., schematic + quasi-separated and quasi-compact; affine) map between algebraic stacks. Then the functor* (1.3)

$$\mathrm{DGSch}_{/\mathcal{Y}',\mathrm{smooth}} \to \mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}$$

defines an equivalence

$$\mathrm{QCoh}(\mathcal{Y}) = \underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y}})^{\mathrm{op}}}{\underleftarrow{lim}} \mathrm{QCoh}(S) \to \underset{(S',g')\in(\mathrm{DGSch}_{/\mathcal{Y}'})^{\mathrm{op}}}{\underleftarrow{lim}} \mathrm{QCoh}(S),$$

and similarly for the $(\mathrm{DGSch}_{\text{qs-qc}})_{/\mathcal{Y}}$ *and* $(\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$ *versions.*

1.2.7. *t-structure.* For any prestack $\mathcal{Y}$, the category $\mathrm{QCoh}(\mathcal{Y})$ has a natural t-structure: an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ is connective (i.e., cohomologically ≤ 0) if its pullback to any scheme is.

Two important features of this t-structure are summarized in the following lemma:

Lemma 1.2.8. *Suppose that $\mathcal{Y}$ is an algebraic stack.*

(a) *The t-structure on* $\mathrm{QCoh}(\mathcal{Y})$ *is compatible with filtered colimits.*[7]

(b) *The t-structure on* $\mathrm{QCoh}(\mathcal{Y})$ *is* left-complete*, i.e., for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$, the natural map*

$$\mathcal{F} \to \underset{n\in\mathbb{N}}{\underleftarrow{lim}}\, \tau^{\geq -n}(\mathcal{F})$$

is an isomorphism, where τ denotes the truncation functor.

(c) *If $f: Z \to \mathcal{Y}$ is a faithfully flat atlas, the functor $f^*: \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(Z)$ is t-exact and conservative.*

We refer the reader to [Lu2, Sect. 1.2.1], for a review of the notion of left-completeness of a t-structure.

For the proof of the lemma, see [GL:QCoh, Cor. 5.2.4]. One first reduces to the case where $\mathcal{Y}$ is an affine DG scheme. In this case $\mathrm{QCoh}(\mathcal{Y})$ is left-complete because it admits a conservative t-exact functor to Vect that commutes with limits, namely, $\Gamma(\mathcal{Y}, -)$.

Remark 1.2.9. Let $\mathcal{Y}$ be an algebraic stack. It is easy to see that the category $\mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$ identifies with $\mathrm{QCoh}({}^{cl}\mathcal{Y})^{\heartsuit}$.

Remark 1.2.10. Suppose again that $\mathcal{Y}$ is classical and algebraic. Suppose in addition that the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is affine. In this case, it is easy to show that $\mathrm{QCoh}(\mathcal{Y})^+$ is canonically equivalent to $D(\mathrm{QCoh}(\mathcal{Y})^{\heartsuit})^+$, see [8] [GL:QCoh, Prop. 5.4.3]. It follows from

[7]By definition, this means that the subcategory $\mathrm{QCoh}(\mathcal{Y})^{>0}$ is preserved under filtered colimits. Note that the subcategory $\mathrm{QCoh}(\mathcal{Y})^{\leq 0}$ automatically has this property.

[8]Here by $D(\mathcal{A})$ for an abelian category $\mathcal{A}$ we mean the canonical DG category, whose homotopy category is the derived category of $\mathcal{A}$, see [Lu2, Sect. 1.3.4].

Lemma 1.2.8 that the entire $\mathrm{QCoh}(\mathcal{Y})$ can be recovered as the left completion of $D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit)$. At least, in characteristic $p > 0$ it can happen that $D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit)$ itself is not left-complete (e.g., A. Neeman [Ne] showed this if $\mathcal{Y}$ is the classifying stack of the additive group over a field of characteristic $p > 0$). However, it is easy to formulate sufficient conditions for $D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit)$ to be left-complete: for example, this happens when $\mathrm{QCoh}(\mathcal{Y})^\heartsuit$ is generated by (every object of $\mathrm{QCoh}(\mathcal{Y})^\heartsuit$ is a filtered colimit of quotients of) objects having finite cohomological dimension. E.g., this tautologically happens when $\mathcal{Y}$ is an affine DG scheme, or more generally, a quasi-projective scheme. From here one deduces that this is also true for any $\mathcal{Y}$ of the form Z/G, where Z is a quasi-projective scheme, and G is an affine algebraic group acting linearly on Z, provided we are in characteristic 0.

Remark 1.2.11. If $\mathcal{Y}$ is an algebraic stack, which is not classical, then for two objects

$$\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{QCoh}(\mathcal{Y})^\heartsuit \simeq \mathrm{QCoh}({}^{cl}\mathcal{Y})^\heartsuit$$

the Exts between these objects computed in $\mathrm{QCoh}(\mathcal{Y})$ and $\mathrm{QCoh}({}^{cl}\mathcal{Y})$ will, of course, be different. [9]

1.3. **Direct images for quasi-coherent sheaves.**

1.3.1. Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism between prestacks. We have a tautologically defined (continuous) functor

$$\pi^* : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1).$$

By the adjoint functor theorem ([Lu1, Cor. 5.5.2.9]), π^* admits a right adjoint, denoted π_*. However, in general, π_* is *not* continuous, i.e., it does *not* commute with colimits.

For $\mathcal{Y} \in \mathrm{PreStk}$ and $p_\mathcal{Y} : \mathcal{Y} \to \mathrm{pt}$ we shall also use the notation

$$\Gamma(\mathcal{Y}, -) := (p_\mathcal{Y})_*.$$

Remark 1.3.2. In fact, π_* defined above, is a pretty "bad" functor. E.g., it does *not* satisfy base change (see Sect. 1.3.3 below for what tis means). Neither does is satisfy the projection formula (see Sect. 1.3.7 for what this means), even for open embeddings. One of the purposes of this paper is to give conditions on π that ensure that π_* is continuous and has other nice properties.

1.3.3. *Base change.* Let $\phi_2 : \mathcal{Y}'_2 \to \mathcal{Y}_2$ be another map of prestacks. Consider the Cartesian diagram

$$\begin{CD}
\mathcal{Y}'_1 @>{\phi_1}>> \mathcal{Y}_1 \\
@V{\pi'}VV @VV{\pi}V \\
\mathcal{Y}'_2 @>{\phi_2}>> \mathcal{Y}_2.
\end{CD}$$

By adjunction, for $\mathcal{F}_1 \in \mathrm{QCoh}(\mathcal{Y}_1)$ we obtain a morphism

$$\phi_2^* \circ \pi_*(\mathcal{F}_1) \to \pi'_* \circ \phi_1^*(\mathcal{F}_1). \tag{1.4}$$

Definition 1.3.4.

(a) *The triple* $(\phi_2, \mathcal{F}_1, \pi)$ *satisfies base change if the map* (1.4) *is an isomorphism.*

(b) *The pair* $(\mathcal{F}_1, \pi)$ *satisfies base change if* (1.4) *is an isomorphism for any* ϕ_2*.*

(c) *The morphism* π *satisfies base change if* (1.4) *is an isomorphism for any* ϕ_2 *and* $\mathcal{F}_1$*.*

---

[9] Sam Raskin points out that the latter observation may serve as an entry point to the world of derived algebraic geometry for those not a priori familiar with it: we start with the abelian category $\mathrm{QCoh}({}^{cl}\mathcal{Y})^\heartsuit$, and the data of $\mathcal{Y}$ encodes a way to promote it to a DG category, namely, $\mathrm{QCoh}(\mathcal{Y})$.

1.3.5. Let us observe the following:

Proposition 1.3.6. *Given* $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$*, for* $\mathcal{F}_1 \in \mathrm{QCoh}(\mathcal{Y}_1)$ *the following conditions are equivalent:*

(i) $(\mathcal{F}_1, \pi)$ *satisfies base change.*

(ii) $(\phi_2, \mathcal{F}_1, \pi)$ *satisfies base change whenever* $\mathcal{Y}_2 = S_2 \in \mathrm{DGSch}^{\mathrm{aff}}$.

(iii) *For any* $S_2' \overset{f_2}{\to} S_2 \overset{g_2}{\to} \mathcal{Y}_2$ *with* $S_2, S_2' \in \mathrm{DGSch}^{\mathrm{aff}}$*, the triple* $(f_2, \mathcal{F}_{S,1}, \pi_S)$ *satisfies base change, where* $\mathcal{F}_{S,1} := \mathcal{F}_1|_{S_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}$ *and* $\pi_S : S_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S_2$.

Proof. Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Suppose that $(\mathcal{F}_1, \pi)$ satisfies (iii). Consider the assignment

$$(S_2 \in (\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}_2}) \rightsquigarrow (\pi_S)_*(\mathcal{F}_{S,1}).$$

The assumption implies that this assignment defines an object $\pi_{*,?}(\mathcal{F}_1) \in \mathrm{QCoh}(\mathcal{Y}_2)$. Moreover, it is easy to see that this object is equipped with a functorial isomorphism

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y}_2)}(\mathcal{F}_2, \pi_{*,?}(\mathcal{F}_1)) \simeq \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y}_1)}(\pi^*(\mathcal{F}_2), \mathcal{F}_1), \quad \mathcal{F}_2 \in \mathrm{QCoh}(\mathcal{Y}_2).$$

Hence, $\pi_{*,?}(\mathcal{F}_1) \simeq \pi_*(\mathcal{F}_1)$, and thus

$$g_2^*(\pi_*(\mathcal{F}_1)) \simeq (\pi_S)_*(\mathcal{F}_{S,1}). \tag{1.5}$$

By the same logic, for any $\phi_2 : \mathcal{Y}_2' \to \mathcal{Y}_2$, and $g_2' : S_2' \to \mathcal{Y}_2'$, we obtain that

$$(g_2')^*(\pi_*' \circ \phi_1^*(\mathcal{F}_1)) \simeq (\pi_{S'})_*(\mathcal{F}_{S',1}),$$

where $\mathcal{F}_{S',1} := \mathcal{F}_1|_{S_2' \underset{\mathcal{Y}_2'}{\times} \mathcal{Y}_1'}$ and $\pi_{S'} : S_2' \underset{\mathcal{Y}_2'}{\times} \mathcal{Y}_1' \to S_2'$. Hence, applying (1.5) to the map

$$g_2' \circ \phi_2 : S_2' \to \mathcal{Y}_2,$$

we obtain

$$(g_2')^*(\pi_*' \circ \phi_1^*(\mathcal{F}_1)) \simeq (\pi_{S'})_*(\mathcal{F}_{S',1}) \simeq (g_2' \circ \phi_2)^*(\pi_*(\mathcal{F}_1)) = (g_2')^*(\phi_2^* \circ \pi_*(\mathcal{F}_1)),$$

as required.

$\square$

1.3.7. *Projection formula.* Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be as above. For $\mathcal{F}_i \in \mathrm{QCoh}(\mathcal{Y}_i)$ by adjunction we have a canonically defined map

$$\mathcal{F}_2 \otimes \pi_*(\mathcal{F}_1) \to \pi_*(\pi^*(\mathcal{F}_2) \otimes \mathcal{F}_1). \tag{1.6}$$

Definition 1.3.8.

(a) *The triple* $(\mathcal{F}_1, \mathcal{F}_2, \pi)$ *satisfies the projection formula if the map* (1.6) *is an isomorphism.*

(b) *The pair* $(\mathcal{F}_2, \pi)$ *satisfies the projection formula if* (1.4) *is an isomorphism for any* $\mathcal{F}_1$.

(c) *The pair* $(\mathcal{F}_1, \pi)$ *satisfies the projection formula if* (1.4) *is an isomorphism for any* $\mathcal{F}_2$.

(d) *The morphism* π *satisfies the projection formula if* (1.4) *is an isomorphism for any* $\mathcal{F}_1$ *and* $\mathcal{F}_2$.

We also give the following definition:

Definition 1.3.9. *The morphism* π *strongly satisfies the projection formula if it satisfies base change and for every* $S_2 \in (\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}_2}$*, the morphism*

$$\pi_S : S_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S_2$$

satisfies the projection formula.

It is easy to see as in Proposition 1.3.6 that if π strongly satisfies the projection formula, then it satisfies the projection formula.

1.3.10. Suppose for a moment that π is schematic, quasi-separated and quasi-compact. In this case, from Proposition 1.3.6, we obtain that π strongly satisfies projection formula.

In particular, for π schematic, quasi-separated and quasi-compact, we obtain the following explicit description of $\pi_*(\mathcal{F}_1)$ for $\mathcal{F}_1 \in \mathrm{QCoh}(\mathcal{Y}_1)$. Namely, for $(S_2, g_2) \in (\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}_2}$, we have

$$g_2^*(\mathcal{F}_2) \simeq (\pi_S)_*(g_1^*(\mathcal{F}_1))$$

for the morphisms as in the following Cartesian diagram

$$\begin{CD}
S_1 @>{g_1}>> \mathcal{Y}_1 \\
@V{\pi'}VV @VV{\pi}V \\
S_2 @>{g_2}>> \mathcal{Y}_2.
\end{CD}$$

Remark 1.3.11. From the above observation for schematic, quasi-separated and quasi-compact morphisms combined with the implication (iii) $\Rightarrow$ (i) in Proposition 1.3.6, we obtain that in Definition 1.3.9, the condition that π should satisfy base change is automatic. Indeed, in the notation of the proof of Proposition 1.3.6, express $(\phi_2)_*(\phi_2^*(-))$ as $(\phi_2)_*(\mathcal{O}_{S_2'})\otimes -$, and similarly for the morphism $S_2' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$.

1.3.12. Assume now that $\mathcal{Y}_2$ is an algebraic stack. Note that any map from an affine (or, more generally, quasi-separated and quasi-compact) DG scheme to $\mathcal{Y}_2$ is schematic, quasi-separated and quasi-compact. This observation reduces the calculation of π_* to one in Sect. 1.3.10. Namely, we have:

Lemma 1.3.13. *Let A be a category mapping to $\mathrm{DGSch}_{/\mathcal{Y}_1}$ (respectively, $(\mathrm{DGSch}_{\text{qs-qc}})_{/\mathcal{Y}_1}$; $(\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}_1}$) as in Sect. 1.2.5. Then for every $\mathcal{F}_1 \in \mathrm{QCoh}(\mathcal{Y}_1)$ we have*

$$\pi_*(\mathcal{F}_1) \simeq \underset{a\in A^{\mathrm{op}}}{\underleftarrow{lim}}\ (\pi \circ g_a)_*(g_a^*(\mathcal{F}_1)).$$

Proof. For any $\mathcal{F}_2 \in \mathrm{QCoh}(\mathcal{Y}_2)$ one has

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y}_1)}(\pi^*(\mathcal{E}_2), \mathcal{F}_1) \simeq \underset{a\in A^{\mathrm{op}}}{\underleftarrow{lim}}\ \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y}_1)}(g_a^* \circ \pi^*(\mathcal{F}_2), g_a^*(\mathcal{F}_1)) \simeq$$
$$\simeq \underset{a\in A^{\mathrm{op}}}{\underleftarrow{lim}}\ \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y}_2)}\left(\mathcal{F}_2, (\pi \circ g_a)_*(g_a^*(\mathcal{F}_1))\right),$$

as required. □

Remark 1.3.14. Inverse limits in $\mathrm{QCoh}(\mathcal{Y})$ exist for formal (i.e., set-theoretical) reasons, see Sect. 0.6.3. We emphasize that they are *not* computed naively, i.e., the value of an inverse limit on S mapping to $\mathcal{Y}$ is not in general isomorphic to the inverse limit of values.

A particularly useful special case of Lemma 1.3.13 is the following:

Corollary 1.3.15. *Suppose that in the situation of Lemma 1.3.13, $\mathcal{Y}_1$ is an algebraic stack, and let $f : Z \to \mathcal{Y}_1$ be an fppf atlas. Let $Z^\bullet/\mathcal{Y}_1$ be its Čech nerve. Consider the morphisms $f^i : Z^i/\mathcal{Y}_1 \to \mathcal{Y}_1$ and set $f^\bullet := \{f^i\}$. Then*

$$\pi_*(\mathcal{F}) \simeq \mathrm{Tot}\left((\pi \circ f^\bullet)_*((f^\bullet)^*(\mathcal{F}))\right). \tag{1.7}$$

1.3.16. *The bounded below part.* Let $\mathrm{QCoh}(\mathcal{Y})^+$ be the bounded below (a.k.a. eventually coconnective) part of $\mathrm{QCoh}(\mathcal{Y})$, i.e.,

$$\mathrm{QCoh}(\mathcal{Y})^+ := \underset{n\in\mathbb{N}}{\cup} \mathrm{QCoh}(\mathcal{Y})^{\geq -n}.$$

We claim:

Corollary 1.3.17. *Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a quasi-separated and quasi-compact morphism between algebraic stacks.*

(a) *The functor*

$$\pi_* : \mathrm{QCoh}(\mathcal{Y}_1)^{\geq -n} \to \mathrm{QCoh}(\mathcal{Y}_2)^{\geq -n}$$

commutes with colimits.

(b) *For any $\mathcal{F}_1 \in \mathrm{QCoh}(\mathcal{Y}_1)^+$, the pair $(\mathcal{F}_1, \pi)$ satisfies base change with respect to morphisms $\mathcal{Y}_2' \to \mathcal{Y}_2$ that are locally of bounded Tor-dimension.*

(c) *For any $\mathcal{F}_1 \in \mathrm{QCoh}(\mathcal{Y}_1)^+$, and for $\mathcal{F}_2 \in \mathrm{QCoh}(\mathcal{Y}_2)^+$* locally of bounded Tor-dimension*, the triple $(\mathcal{F}_1, \mathcal{F}_2, \pi)$ satisfies the projection formula.*

Proof. As in Proposition 1.3.6, it is easy to see that we can assume that $\mathcal{Y}_2 = S$ is an affine DG scheme. [10]

To prove point (a), it suffices to show that for each $i \in \mathbb{Z}$ the functor

$$H^i(\pi_*) : \mathrm{QCoh}(\mathcal{Y}_1)^{\geq -n} \to \mathrm{QCoh}(S)^{\heartsuit}$$

commutes with filtered colimits. By assumption, $\mathcal{Y}_1$ is quasi-compact; hence it admits an atlas $f : Z \to \mathcal{Y}_1$ with Z being an affine DG scheme. (In fact, all we need for the argument below is that f be quasi-separated and quasi-compact.) Since $\mathcal{Y}_1$ is quasi-separated, we obtain that all the terms of the Čech nerve $Z^\bullet/\mathcal{Y}_1$ are also quasi-separated and quasi-compact.

Let us apply Corollary 1.3.15. The functors $(\pi \circ f^i)_* \circ (f^i)^*$ from the RHS of (1.7) commute with colimits because the morphisms $\pi \circ f^i : Z^i \to \mathcal{Y}_2$ are schematic, quasi-separated and quasi-compact. So for each $m \in \mathbb{N}$ the functor

$$\mathcal{F} \mapsto \mathrm{Tot}_{\leq m}\left((\pi \circ f^\bullet)_*((f^\bullet)^*(\mathcal{F}))\right)$$

commutes with colimits. But if $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_1)^{\geq -n}$ then for each $m > i + n$ the morphism $\mathrm{Tot} \to \mathrm{Tot}_{\leq m}$ induces an isomorphism

$$H^i(\mathrm{Tot}_{\leq m}\left((\pi \circ f^\bullet)_*((f^\bullet)^*(\mathcal{F}))\right)) \xrightarrow{\sim} H^i(\mathrm{Tot}\left((\pi \circ f^\bullet)_*((f^\bullet)^*(\mathcal{F}))\right)).$$

So $H^i(\pi_*) : \mathrm{QCoh}(\mathcal{Y}_1)^{\geq -n} \to \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$ commutes with filtered colimits.

Points (b) and (c) of the proposition follow similarly. □

1.4. **Statements of the results on** $\mathrm{QCoh}(\mathcal{Y})$**.**

[10]For point (a) we are using the fact that in a limit of DG categories $\underset{\overleftarrow{i}}{lim}\, \mathbf{C}_i$, where the transition functors are continuous, colimits of objects are calculated component-wise.

1.4.1. *The main result.* The following theorem will be proved in Sect. 2:

Theorem 1.4.2. *Let $\mathcal{Y}$ be a QCA algebraic stack. Then*

(i) *The functor* $\mathcal{F} \mapsto \Gamma(\mathcal{Y}, \mathcal{F}) : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}$ *is continuous (i.e., it commutes with colimits, equivalently, with filtered colimits, and equivalently, with infinite direct sums);*

(ii) *There exists an integer n (that depends only on $\mathcal{Y}$) such that $H^i(\Gamma(\mathcal{Y}, \mathcal{F})) = 0$ for all $i > n$ and all $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$.*

Note that statement (i) can be rephrased as follows: if $\mathcal{Y}$ is a QCA algebraic stack, then the object $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ is compact.

Corollary 1.4.3. *Let $\mathcal{Y}$ be a QCA algebraic stack. Then an object of* $\mathrm{QCoh}(\mathcal{Y})$ *is compact if and only if it is perfect.*

We recall that an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ is called *perfect* if its pullback to any affine DG scheme is perfect. By [GL:QCoh, Lemma 4.2.2], this is equivalent to $\mathcal{F}$ being dualizable in $\mathrm{QCoh}(\mathcal{Y})$, regarded as a monoidal category.

Proof. If $\mathcal{F}$ is perfect the functor $\mathcal{M}aps_{\mathrm{QCoh}}(\mathcal{F}, -)$ can be rewritten as $\Gamma(\mathcal{Y}, \mathcal{F}^* \otimes -)$, so it is continuous by Theorem 1.4.2(i).

On the other hand, for any algebraic stack $\mathcal{Y}$, any compact object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ is perfect. Indeed, let S be an affine DG scheme equipped with a morphism $f : S \to \mathcal{Y}$, then the object $f^*(\mathcal{F}) \in \mathrm{QCoh}(S)$ is compact (because its right adjoint f_* is continuous), so $f^*(\mathcal{F})$ is perfect (see, e.g., [BFN, Lemma 3.4]). $\square$

1.4.4. *A relative version.*

Corollary 1.4.5. *Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a QCA morphism between prestacks.*

(i) *The functor $\pi_* : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2)$ is continuous and strongly satisfies the projection formula.*

(ii) *If $\mathcal{Y}_2$ is a quasi-compact algebraic stack,* [11] *there exists n such that π_* maps $\mathrm{QCoh}(\mathcal{Y}_1)^{\leq 0}$ to $\mathrm{QCoh}(\mathcal{Y}_2)^{\leq n}$.*

Proof. To prove point (i), by definition, it suffices to consider the case when $\mathcal{Y}_2$ is an affine DG scheme.

In this case the continuity of π_* follows immediately from Theorem 1.4.2(i). Indeed, the functor

$$\Gamma(\mathcal{Y}_2, -) : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{Vect}$$

is continuous and conservative, and $\Gamma(\mathcal{Y}_2, -) \circ \pi_* \simeq \Gamma(\mathcal{Y}_1, -)$, so the continuity of $\Gamma(\mathcal{Y}_1, -)$ implies that for π_*.

The fact that π satisfies the projection formula follows formally from the continuity of π_* (express $\mathcal{F}_2$ as a colimit of copies of the structure sheaf). By Remark 1.3.11, the projection formula implies base change.

To prove point (ii), it is again sufficient to do so after base changing by means of an fppf map $S_2 \to \mathcal{Y}_2$, where S_2 is an affine DG scheme. In this case, the assertion follows from Theorem 1.4.2(ii).

$\square$

[11] Or, more generally, if there exists a map $f : Z \to \mathcal{Y}_2$, where Z is an affine DG scheme, and f is a surjection in the faithfully flat topology (see, e.g., [GL:Stacks, Sect. 2.3.1], where the notion of surjectivity is recalled).

1.4.6. *Generation by the heart.* Let $\mathcal{Y}$ be an algebraic stack.

Definition 1.4.7. *We say that $\mathcal{Y}$ is n-*coconnective *if the object $\mathcal{O}_\mathcal{Y} \in \mathrm{QCoh}(\mathcal{Y})$ belongs to $\mathrm{QCoh}(\mathcal{Y})^{\geq -n}$.*

Definition 1.4.8. *We say that $\mathcal{Y}$ is* eventually coconnective *if it is n-coconnective for some n; equivalently, if $\mathcal{O}_\mathcal{Y}$ is bounded below, i.e., is eventually coconnective as an object of $\mathrm{QCoh}(\mathcal{Y})$.*

Remark 1.4.9. The notion of n-connectivity makes sense for all prestacks, and not just algebraic stacks, see [GL:Stacks, Sect. 2.4.7]. The fact that the two notions coincide for algebraic stacks is established in [GL:Stacks, Proposition 4.6.4].

The stratification technique used in the proof of Theorem 1.4.2 also allows to prove the following result (see Sect. 2.6):

Theorem 1.4.10. *Suppose that an algebraic stack $\mathcal{Y}$ is QCA and eventually coconnective. Then $\mathrm{QCoh}(\mathcal{Y})$ is generated by $\mathrm{QCoh}(\mathcal{Y})^\heartsuit$.*

Corollary 1.4.11. *Let $\mathcal{Y}$ be a QCA algebraic stack, which is eventually coconnective, and such that the underlying classical stack ${}^{cl}\mathcal{Y}$ is Noetherian. Then the category $\mathrm{QCoh}(\mathcal{Y})$ is generated by $\mathrm{Coh}(\mathcal{Y})^\heartsuit$.*

This follows from Theorem 1.4.10 and the following fact [LM, Corollary 15.5]: every object of $\mathrm{QCoh}(\mathcal{Y})^\heartsuit$ is a union of its coherent sub-objects.

1.4.12. *Other results.* We will also prove Theorem 4.3.1, which says, among other things, that in the situation of Corollary 1.4.11 one has $\mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}') = \mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}')$ for any prestack $\mathcal{Y}'$.

2. Proof of Theorems 1.4.2 and 1.4.10

The proof of Theorem 1.4.2 occupies Sects. 2.1-2.5. Theorem 1.4.10 is proved in Sect. 2.6.

2.1. **Reducing the statement to a key lemma.**

2.1.1. *Reducing statement (i) to statement (ii).* Let $\alpha \mapsto \mathcal{F}_\alpha$ be a collection of objects of $\mathrm{QCoh}(\mathcal{Y})$. We need to show that for any $i \in \mathbb{Z}$ the natural map

$$\bigoplus_\alpha H^i(\Gamma(\mathcal{Y}, \mathcal{F}_\alpha)) \to H^i(\Gamma(\mathcal{Y}, \bigoplus_\alpha \mathcal{F}_\alpha))$$

is an isomorphism. Suppose we have proved Theorem 1.4.2(ii), i.e., there exists n such that the functor $H^i\left(\Gamma(\mathcal{Y}, -)\right)$ vanishes on $\mathrm{QCoh}(\mathcal{Y})^{<-i-n}$. Then

$$H^i(\Gamma(\mathcal{Y}, \mathcal{F}_\alpha)) = H^i(\Gamma(\mathcal{Y}, \tau^{\geq -i-n-1}(\mathcal{F}_\alpha))),$$
$$H^i(\Gamma(\mathcal{Y}, \underset{\alpha}{\oplus}\mathcal{F}_\alpha)) = H^i(\Gamma(\mathcal{Y}, \tau^{\geq -i-n-1}(\underset{\alpha}{\oplus}\mathcal{F}_\alpha))).$$

Since the t-structure on $\mathrm{QCoh}(\mathcal{Y})$ is compatible with filtered colimits (see Lemma 1.2.8(a)), the morphism

$$\bigoplus_\alpha \tau^{\geq -i-n-1}(\mathcal{F}_\alpha) \to \tau^{\geq -i-n-1}\left(\bigoplus_\alpha \mathcal{F}_\alpha\right).$$

is an isomorphism. So we have to prove that the morphism

$$\bigoplus_\alpha H^i(\Gamma(\mathcal{Y}, \tau^{\geq -i-n-1}(\mathcal{F}_\alpha))) \to H^i(\Gamma(\mathcal{Y}, \bigoplus_\alpha \tau^{\geq -i-n-1}(\mathcal{F}_\alpha)))$$

is an isomorphism. We have $\tau^{\geq -i-n-1}(\mathcal{F}_\alpha) \in \mathrm{QCoh}(\mathcal{Y})^{\geq r}$, where $r = -i-n-1$. Now Theorem 1.4.2(i) follows from Corollary 1.3.17.

□

2.1.2. *Reducing statement (ii) to a key lemma.*

Lemma 2.1.3. *Let $n \in \mathbb{Z}$. Suppose that for any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$ we have*

$$H^i\left(\Gamma(\mathcal{Y}, \mathcal{F})\right) = 0 \ \textit{ for } \ i > n. \tag{2.1}$$

Then (2.1) *holds for any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$.*

Proof. The statement is clear if $\mathcal{F}$ is bounded below. To treat the general case, recall that the t-structure on $\mathrm{QCoh}(\mathcal{Y})$ is left-complete, see Lemma 1.2.8(b). Since the functor $\Gamma(\mathcal{Y}, -) \simeq \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{O}_{\mathcal{Y}}, -)$ commutes with inverse limits this implies that

$$\Gamma(\mathcal{Y}, \mathcal{F}) = \underset{\underset{m}{\longleftarrow}}{lim}\, \Gamma(\mathcal{Y}, \tau^{\geq -m}(\mathcal{F})).$$

If $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{<0}$ then the complexes $\Gamma(\mathcal{Y}, \tau^{\geq -m}(\mathcal{F}))$ are concentrated in degrees $< n$. Since the functor $\underset{\underset{m}{\longleftarrow}}{lim}$ in Vect has cohomological amplitude $[0, 1]$ we see that $\Gamma(\mathcal{Y}, \mathcal{F})$ is concentrated in degrees $\leq n$. So (2.1) holds for any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{<0}$. Therefore it holds for any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$: use the exact triangle $\tau^{<0}(\mathcal{F}) \to \mathcal{F} \to \tau^{\geq 0}(\mathcal{F})$. $\square$

By Lemma 2.1.3, to prove Theorem 1.4.2(ii) it suffices to prove the following key lemma.

Lemma 2.1.4. *Let $\mathcal{Y}$ be a QCA stack. Then there exists an integer $n_{\mathcal{Y}}$ such that for any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$ we have*

$$H^i\left(\Gamma(\mathcal{Y}, \mathcal{F})\right) = 0 \ \textit{ for } \ i > n_{\mathcal{Y}}\,.$$

The lemma will be proved in Sects. 2.2-2.5.

2.2. **Easy reduction steps.**

2.2.1. *Reduction to the classical case.* Let ${}^{cl}\mathcal{Y} \overset{{}^{cl}i}{\hookrightarrow} \mathcal{Y}$ be the embedding of the classical stack underlying $\mathcal{Y}$. Any $\mathcal{F}$ as in the Lemma 2.1.4 belongs to the essential image of the functor ${}^{cl}i_*$. Since ${}^{cl}i_*$ is t-exact, we can replace the original $\mathcal{Y}$ by ${}^{cl}\mathcal{Y}$, with the same estimate for n.

So for the rest of this section we will assume that $\mathcal{Y}$ is classical.

2.2.2. *Reduction to the case when $\mathcal{Y}$ is reduced.* Let $\mathcal{Y}_{red} \overset{i_{red}}{\hookrightarrow} \mathcal{Y}$ be the corresponding reduced substack.

Any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$ admits an increasing filtration with subquotients belonging to the essential image of the functor $(i_{red})_*$. Since the functor

$$H^i\left(\Gamma(\mathcal{Y}, -)\right) : \mathrm{QCoh}(\mathcal{Y})^{\heartsuit} \to \mathrm{Vect}^{\heartsuit}$$

commutes with filtered colimits (by Corollary 1.3.17(a)), by the same logic as above, we can replace $\mathcal{Y}$ by $\mathcal{Y}_{red}$ with the same estimate on $n_{\mathcal{Y}}$.

So we can assume that $\mathcal{Y}$ is reduced.

2.3. **Devissage.**

2.3.1. We begin with the following observation.

Let $\mathcal{X} \overset{\imath}{\hookrightarrow} \mathcal{Y}$ be a closed substack and $\overset{\circ}{\mathcal{Y}} \overset{\jmath}{\hookrightarrow} \mathcal{Y}$ the complementary open substack, such that the map $\jmath$ is quasi-compact. Let $d \in \mathbb{Z}$ be such that the functor $\jmath_*$ has cohomological amplitude $\leq d$ (it exists because $\mathcal{Y}$ itself is quasi-compact).

Lemma 2.3.2. *If Lemma 2.1.4 holds for $\mathcal{X}$ and $\overset{\circ}{\mathcal{Y}}$ then it holds for $\mathcal{Y}$ with*

$$n_{\mathcal{Y}} := \max(n_{\overset{\circ}{\mathcal{Y}}}, n_{\mathcal{X}} + d + 1).$$

Proof. For $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$ consider the exact triangle

$$\mathcal{F}' \to \mathcal{F} \to \jmath_* \circ \jmath^*(\mathcal{F}),$$

where $\mathcal{F}'$ is *set-theoretically* supported on $\mathcal{X}$.

It is enough to show that

$$H^r\left(\Gamma(\mathcal{Y}, \jmath_* \circ \jmath^*(\mathcal{F}))\right) = 0 \ \text{ for } \ r > n_{\mathcal{Y}}\,, \tag{2.2}$$

$$H^r\left(\Gamma(\mathcal{Y}, \mathcal{F}')\right) = 0 \ \text{ for } \ r > n_{\mathcal{Y}}\,. \tag{2.3}$$

The vanishing in (2.2) is clear because $\Gamma(\mathcal{Y}, \jmath_* \circ \jmath^*(\mathcal{F})) \simeq \Gamma(\overset{\circ}{\mathcal{Y}}, \jmath^*(\mathcal{F}))$ and $n_{\mathcal{Y}} \geq n_{\overset{\circ}{\mathcal{Y}}}$.

Let us prove (2.3). Note that $\mathcal{F}'$ has finitely many cohomology sheaves and all of them are in degrees $\leq d+1$. We have $n_{\mathcal{Y}} \geq n_{\mathcal{X}} + d + 1$. So to prove (2.3) it suffices to show that if a sheaf $\mathcal{F}'' \in \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$ is set-theoretically supported on $\mathcal{X}$ then

$$H^r\left(\Gamma(\mathcal{Y}, \mathcal{F}'')\right) = 0 \ \text{ for } \ r > n_{\mathcal{X}}\,. \tag{2.4}$$

Represent $\mathcal{F}''$ as a filtered colimit of sheaves $\mathcal{F}''_\alpha$ so that each $\mathcal{F}''_\alpha$ admits a finite filtration with subquotients belonging to the essential image of $\imath_* : \mathrm{QCoh}(\mathcal{X})^{\heartsuit} \to \mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$. By assumption, for each α and each $r > n_{\mathcal{X}}$ one has $H^r\left(\Gamma(\mathcal{Y}, \mathcal{F}''_\alpha)\right) = 0$. So (2.4) follows from Corollary 1.3.17. □

2.3.3. By the above, we can assume that $\mathcal{Y}$ is reduced. The next proposition is valid over any ground field.

Proposition 2.3.4. *There exists a finite decomposition of $\mathcal{Y}$ into a union of locally closed reduced algebraic substacks $\mathcal{Y}_i$, each of which satisfies:*

- *The locally closed embedding $\mathcal{Y}_i \hookrightarrow \mathcal{Y}$ is quasi-compact;*
- *There exists a finite surjective flat morphism $\pi : \mathcal{Z}_i \to \mathcal{Y}_i$ with $\mathcal{Z}_i$ being a quotient of a quasi-separated and quasi-compact scheme Z_i by an action of an affine algebraic group (of finite type) over k. Moreover:*

  (i) One can arrange so that Z_i are quasi-projective over an affine scheme, and the group action is linear with respect to this projective embedding.

  (ii) If $\mathrm{char}\, k = 0$, π can be chosen to be étale.

This proposition will be proved in Sect. 2.5.

Remark 2.3.5. Point (i) of the proposition will be used in the proof of Theorem 1.4.10 but not in the proof of Lemma 2.1.4.

We are now going to deduce Lemma 2.1.4 for $\mathcal{Y}$ as above from Proposition 2.3.4.

2.3.6. By induction and Lemma 2.3.2, it is enough to prove Lemma 2.1.4 for the algebraic stacks $\mathcal{Y}_i$.

2.3.7. Let $\mathcal{Z}_i \to \mathcal{Y}_i$ be a finite surjective étale morphism as in Proposition 2.3.4. We claim that if Lemma 2.1.4 holds for $\mathcal{Z}_i$ then it holds for $\mathcal{Y}_i$.

To see this, note that any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_i)$ is a direct summand of $\pi_* \circ \pi^*(\mathcal{F}) = \mathcal{F} \otimes \pi_*(\mathcal{O}_{\mathcal{Z}_i})$ (use the trace morphism $\pi_*(\mathcal{O}_{\mathcal{Z}_i}) \to \mathcal{O}_{\mathcal{Y}_i}$).

(Note that the last manipulation used the $\mathrm{char}\, k = 0$ assumption. But this is not the most crucial place where we will use it.)

Thus, it is sufficient to prove Lemma 2.1.4 for a stack $\mathcal{Z}$ of the form Z/G, where Z is a quasi-separated and quasi-compact scheme, and G is an affine algebraic group of finite type over k.

2.4. **Quotients of schemes by algebraic groups.** Let G be a reductive algebraic group over k. Consider the stack $BG := \mathrm{pt}/G$.

Lemma 2.4.1. *The functor*

$$\Gamma(BG, -) : \mathrm{QCoh}(BG) \to \mathrm{Vect}$$

is t-exact. More precisely,

$$H^i(\Gamma(BG, M)) = (H^i(M))^G, \qquad M \in \mathrm{QCoh}(BG). \tag{2.5}$$

It is here that we use the characteristic 0 assumption.

Proof. By Remark 1.2.10 (which relies on [GL:QCoh, Prop. 5.4.3]), $\mathrm{QCoh}(BG)$ is the left completion of $D(\mathcal{A})$, where $\mathcal{A} := \mathrm{QCoh}(BG)^\heartsuit$ is the abelian category of G-modules. But $\mathcal{A}$ is semisimple (because $\mathrm{char}\, k = 0$), so $D(\mathcal{A})$ is left-complete and $\mathrm{QCoh}(BG) = D(\mathcal{A})$. The lemma follows. □

Remark 2.4.2. In the proof of Lemma 2.4.1 we used [GL:QCoh, Prop. 5.4.3]. Instead, one can argue as follows. By Lemma 2.1.3 and Corollary 1.3.17, it suffices to prove (2.5) if $M \in \mathcal{A} := \mathrm{QCoh}(BG)^\heartsuit$. In this case applying Corollary 1.3.15 to the atlas $\mathrm{pt} \to BG$ we see that $H^i(\Gamma(BG, M))$ identifies with the usual $H^i(G, M)$, and the latter is isomorphic to $\mathrm{Ext}^i_{\mathcal{A}}(k, M)$. It remains to use the semisimplicity of $\mathcal{A}$.

Lemma 2.4.3. *Let Z be a quasi-separated and quasi-compact scheme equipped with an action of an affine algebraic group G. Then Lemma 2.1.4 holds for $\mathcal{Z} = Z/G$.*

Proof. The canonical morphism $f : \mathcal{Z} \to BG$ is schematic, quasi-separated and quasi-compact. Embed G into a reductive group G' and let f' be the composition $\mathcal{Z} \xrightarrow{f} BG \to BG'$. Then f' is still schematic, quasi-separated and quasi-compact, so the cohomological amplitude of the functor $f'_* : \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{Vect}$ is bounded above. On the other hand, the functor

$$\Gamma : \mathrm{QCoh}(BG') \to \mathrm{Vect}$$

is t-exact by Lemma 2.4.1. □

2.5. **Proof of Proposition 2.3.4.** This will conclude the proof of Lemma 2.1.4 in view of Sect. 2.3.

2.5.1. The proof of the proposition is based on the following lemma.

Lemma 2.5.2. *Let $\mathcal{Y} \neq \emptyset$ be a classical algebraic stack, which is quasi-compact and whose inertia stack is of finite presentation over $\mathcal{Y}$. Then there exists a finite decomposition of $\mathcal{Y}$ into a union of locally closed reduced algebraic substacks $\mathcal{Y}_i$, each of which satisfies:*

- *The locally closed embedding $\mathcal{Y}_i \hookrightarrow \mathcal{Y}$ is quasi-compact;*
- *Each $\mathcal{Y}_i$ admits a map $\varphi_i : \mathcal{Y}_i \to X'_i$, where X'_i is an affine scheme with the following property:*

  There exists a finite fppf morphism $f_i : X_i \to X'_i$, and a flat group-scheme of finite presentation $\mathcal{G}_i$ over X_i such that $X_i \underset{X'_i}{\times} \mathcal{Y}_i$ is isomorphic to the classifying stack $B\mathcal{G}_i$.

  Moreover, we can always arrange so that X_i and X'_i are integral. In the characteristic 0 case, one can choose f_i to be étale.

Proof. We are going to apply [LM, Theorem 11.5]. We note that in *loc.cit.*, it is stated under the assumption that $\mathcal{Y}$ is Noetherian. However, the only place where the Noetherian hypothesis is used in the proof is to ensure that the inertia stack be of finite presentation over $\mathcal{Y}$, which is what we are imposing by assumption.

The above theorem yields a decomposition of $\mathcal{Y}$ as in the lemma, with the only difference that the morphisms

$$f_i : X_i \to X'_i$$

are just fppf. We have to show that each X'_i admits a finite decomposition into a union of locally closed integral subschemes $X'_{i,j}$, each of which satisfies:

- The locally closed embeddings $X'_{i,j} \hookrightarrow X'_i$ are quasi-compact;
- For every j, there exists a finite fppf map $g_{i,j} : \widetilde{X}'_{i,j} \to X'_{i,j}$, such that f_i admits a section after a base change by $g_{i,j}$.

  Moreover, the schemes $\widetilde{X}'_{i,j}$ can be chosen integral. In the characteristic 0 case, $g_{i,j}$ can be chosen étale.

We claim, however, that this is the case for any fppf map $f : X \to X'$ between reduced affine schemes. Indeed, recall that whenever $f : X \to X'$ is an fppf morphism of schemes with X' affine, we can always realize it as a base change

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X^0 \arrow[d, "f^0"] \\ X' \arrow[r] & X'^0, \end{tikzcd}$$

where $f^0 : X^0 \to X'^0$ is an fppf morphisms of schemes of finite type over k. Hence, our assertion reduces to the case when X' is of finite type.

In the latter case, by Noetherian induction it is enough to show that it contains a non-empty open subset $\overset{\circ}{X}{}'$ with a finite flat (in characteristic 0, étale) cover $g : \widetilde{X} \to \overset{\circ}{X}{}'$, such that f admits a section after a base change by g.

Let K' denote the field of fractions of X'. Clearly, X has a point over some finite extension $\widetilde{K}'$ of K'.

Taking $\widetilde{X}'$ to be any integral scheme finite over X' with field of fractions $\widetilde{K}'$, we obtain that the map $\widetilde{X}' \to X$ is well-defined over some non-empty open subset $\overset{\circ}{X}{}' \subset X'$, as required.

Moreover in characteristic 0, the map $\widetilde{X}' \to X'$ is generically étale over X', since $\widetilde{K}'/K'$ is separable. $\square$

Proof of Proposition 2.3.4. Let $\mathcal{Y}_i$, X_i', X_i, and $\mathcal{G}_i$ be as in Lemma 2.5.2. Note that for each field-valued point of X_i, the fiber of $\mathcal{G}_i$ at it identifies with the group of automorphisms of the corresponding point of $\mathcal{Y}_i$. Therefore, by the QCA condition, all these groups are affine.

As the index i will be fixed, for the rest of the proof, we shall suppress it from the notation.

It is sufficient to show that X' admits a finite decomposition into a union of locally closed reduced subschemes X_l', each of which satisfies:

- The locally closed embedding $X_l' \hookrightarrow X'$ is quasi-compact;
- The stack
$$\mathcal{Z}_l := B\mathcal{G} \underset{X'}{\times} X_l'$$
(which is tautologically the same as $(X \underset{X'}{\times} \mathcal{Y}) \underset{\mathcal{Y}}{\times} (\mathcal{Y} \underset{X'}{\times} X_l')$, viewed as equipped with a map to $\mathcal{Y} \underset{X'}{\times} X_l')$, is isomorphic to a stack of the form Z_l/G_l, where Z_l is a quasi-separated and quasi-compact scheme, and G_l is an affine algebraic group of finite type over k. Moreover, Z_l can be chosen to be quasi-projective over an affine scheme, and the action of G_l on it linear with respect this projective embedding.

Since $\mathcal{G}$ and X are of finite presentation over X', they come by base change from a map $X' \to X'^0$, where X'^0 is of finite type over k. Hence, is is enough to prove the assertion in the case when X' (and hence X and $\mathcal{G}$) are of finite type.

In the latter case, by Noetherian induction, it is sufficient to find a non-empty open subset $\overset{\circ}{X}{}' \subset X'$, such that $B\mathcal{G} \underset{X'}{\times} \overset{\circ}{X}{}'$ is of the form Z/G specified above. Moreover, since the morphism $X \to X'$ is finite, it is sufficient to find the corresponding open $\overset{\circ}{X}$ in X.

Recall that X was assumed integral. Let K be the field of fractions of X. Let
$$\mathcal{G}_K := \mathcal{G} \underset{X}{\times} \mathrm{pt}$$
be the corresponding algebraic group over K. Since $\mathcal{G}_K$ is affine, we can embed it into $GL(n)_K := GL(n) \times \mathrm{pt}$.

By Chevalley's theorem,
$$Z_K := GL(n)_K/\mathcal{G}_K$$
is a quasi-projective scheme over K equipped with a linear action of $GL(n)$.

Hence, there exists a non-empty open subscheme $\overset{\circ}{X} \subset X$, such that $\mathcal{G}|_{\overset{\circ}{X}}$ admits a map into $GL(n) \times \overset{\circ}{X}$, and the stack-theoretic quotient
$$(GL(n) \times \overset{\circ}{X})/(\mathcal{G}|_{\overset{\circ}{X}})$$
is isomorphic to a quasi-projective scheme Z over $\overset{\circ}{X}$, and moreover the natural action of $GL(n)$ on it is linear.

Thus, $B\mathcal{G}|_{\overset{\circ}{X}} \simeq Z/GL(n)$, as required.

$\square$

2.6. **Proof of Theorem 1.4.10.** Below we give a direct proof. In the case when $\mathcal{Y}$ is locally almost of finite type, one can deduce Theorem 1.4.10 from Proposition 3.5.1, as explained in Remark 3.5.2.

2.6.1. *Reduction to the reduced classical case.* Let ${}^{cl}\mathcal{Y} \overset{{}^{cl}i}{\hookrightarrow} \mathcal{Y}$ be the embedding of the classical stack underlying $\mathcal{Y}$. We claim that $\mathrm{QCoh}(\mathcal{Y})$ is generated by the essential image of the functor ${}^{cl}i_*$. To see this, use the filtration of $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ by objects $\mathcal{F} \otimes \tau^{\leq -n}(\mathcal{O}_{\mathcal{Y}})$, $n \in \mathbb{Z}_+$, which is finite by the eventual coconnectivity assumption.

So without loss of generality we can assume that $\mathcal{Y}$ is classical. A similar argument allows to assume that $\mathcal{Y}$ is reduced.

2.6.2. Using Proposition 2.3.4, the statement of the theorem results from the combination of the following three lemmas:

Lemma 2.6.3. *Let Z be a quasi-projective scheme equipped with a linear action of an affine algebraic group G. Then $\mathrm{QCoh}(Z/G)$ is generated by the heart of its t-structure.*

Lemma 2.6.4. *If $\mathcal{Z} \to \mathcal{Y}$ is a finite étale map, and $\mathrm{QCoh}(\mathcal{Z})$ is generated by the heart of its t-strcuture, then the same is true for $\mathrm{QCoh}(\mathcal{Y})$.*

Lemma 2.6.5. *In the situation of Lemma 2.3.2, if both $\mathrm{QCoh}(\overset{\circ}{\mathcal{Y}})$ and $\mathrm{QCoh}(\mathcal{X})$ are generated by the hearts of their t-structures, then the same is true for $\mathrm{QCoh}(\mathcal{Y})$.*

2.6.6. *Proof of Lemma 2.6.3.* It is easy to see that $\mathrm{QCoh}(Z/G)$ is generated by objects of the form $\mathcal{O}_Z(-i)$, where $\mathcal{O}_Z(1)$ denotes the corresponding ample line bundle on Z.

□

2.6.7. *Proof of Lemma 2.6.4.* This follows from the fact that every object $\mathcal{F} \in \mathcal{O}(\mathcal{Y})$ is a direct summand of $\pi_* \circ \pi^*(\mathcal{F})$, see Sect. 2.3.7.

□

2.6.8. *Proof of Lemma 2.6.5.* Let $\mathrm{QCoh}(\mathcal{Y})^{\spadesuit} \subset \mathrm{QCoh}(\mathcal{Y})$ be the subcategory generated by $\mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$. The subcategory $\mathrm{QCoh}(\mathcal{Y})^{\spadesuit}$ contains the essential images of the functors

$$\jmath_* : \mathrm{QCoh}(\overset{\circ}{\mathcal{Y}}) \to \mathrm{QCoh}(\mathcal{Y}), \quad \imath_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$$

because Theorem 1.4.10 holds for $\overset{\circ}{\mathcal{Y}}$ and $\mathcal{X}$, and the above functors have bounded cohomological amplitude. We have to show that each $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ belongs to $\mathrm{QCoh}(\mathcal{Y})^{\spadesuit}$.

Consider the exact triangle

$$(\mathcal{O}_{\mathcal{Y}})_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}} \to \jmath_* \circ \jmath^*(\mathcal{O}_{\mathcal{Y}}), \tag{2.6}$$

where

$$(\mathcal{O}_{\mathcal{Y}})_{\mathcal{X}} := \mathrm{Cone}\,(\mathcal{O}_{\mathcal{Y}} \to \jmath_* \circ \jmath^*(\mathcal{O}_{\mathcal{Y}}))\,[-1].$$

The object $(\mathcal{O}_{\mathcal{Y}})_{\mathcal{X}}$ is bounded, and each of its cohomologies admits a filtration with subquotients that lies in the essential image of $\imath_*$. Hence, for any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$, the object $\mathcal{F} \otimes H^i((\mathcal{O}_{\mathcal{Y}})_{\mathcal{X}})$ also admits a filtration with subquotients (i.e., the cones of the maps of one term of the filtration into the next) that lie in the essential image of $\imath_*$. In particular, $\mathcal{F} \otimes (\mathcal{O}_{\mathcal{Y}})_{\mathcal{X}} \in \mathrm{QCoh}(\mathcal{Y})^{\spadesuit}$.

Tensoring (2.6) by $\mathcal{F}$, we obtain an exact triangle

$$\mathcal{F} \otimes (\mathcal{O}_{\mathcal{Y}})_{\mathcal{X}} \to \mathcal{F} \to \jmath_* \circ \jmath^*(\mathcal{F}),$$

which implies our assertion.

□

Remark 2.6.9. If $\mathcal{Y}$ is locally Noetherian and $\mathcal{F}$ is perfect, then the object $\mathcal{F}\otimes(\mathcal{O}_\mathcal{Y})_\mathcal{X}$ is isomorphic to

$$\underset{n}{\underset{\longrightarrow}{lim}}\,(\imath_n)_*\circ\imath_n^!(\mathcal{F}),$$

where $\imath_n$ denotes the embedding of the n-th infinitesimal neighborhood of $\mathcal{X}$. This is not necessarily true without the perfectness condition. In general, the !-pullback functor is "bad" (no continuity, no commutation with base change), just like the $*$-pushforward with respect to a non-quasi-compact morphism (see Sect. 1.3.1).

However, this state of affairs with the !-pullback functor can be remedied by replacing the category QCoh($\mathcal{Y}$) by IndCoh($\mathcal{Y}$), considered in the next section.

3. Implications for ind-coherent sheaves

This and the next section are concerned with the category IndCoh on algebraic stacks and, more generally, prestacks. As was mentioned in the introduction, IndCoh is another natural paradigm for "sheaf theory" on stacks.

However, the reader, who is only interested in applications to D-modules, may skip these two sections. Although it is more natural to connect D-modules to the category IndCoh, it will be indicated in Sect. 6.3.17 that if our algebraic stack is eventually coconnective, one can bypass IndCoh, and relate D-mod to QCoh directly. The only awkwardness that will occur is the relation between Verdier duality on coherent D-modules and Serre duality on coherent sheaves, the latter being more naturally interpreted within IndCoh rather than QCoh.

The material in this section is organized as follows. In Sect. 3.1 we recall the condition of being "locally almost of finite type". In Sect. 3.2 we recall the basic facts about the category IndCoh. In Sects. 3.3-3.5 we prove the compact generation and describe the category of compact objects of IndCoh on a QCA algebraic stack. In Sect. 3.6 we introduce the functor of direct image on IndCoh for maps between QCA algebraic stacks.

3.1. **The "locally almost of finite type" condition.** Unlike QCoh, the category IndCoh (and also D-mod, considered later in the paper) only makes sense on (pre)stacks that satisfy a certain finite-typeness hypothesis, called "locally almost of finite type".

For general prestacks this condition may seem as too technical (we review it below). It does appear simpler when applied to algebraic stacks. The reader will not lose much by considering only those prestacks that are algebraic stacks; all the new results in this paper that concern IndCoh and D-mod are about algebraic stacks.

We shall nevertheless, discuss IndCoh in the framework of arbitrary prestacks locally almost of finite type, because this seems to be the natural level of generality.

3.1.1. An affine DG scheme Spec(A) is said to be almost of finite type over k if

- $H^0(A)$ is a finitely generated algebra over k.
- Each $H^{-i}(A)$ is finitely generated as a module over $H^0(A)$.

The property of being almost of finite type is local with respect to Zariski topology. A DG scheme Z is said to be locally of almost finite type if it can be covered by affines, each of which is almost of finite type. Equivalently, Z is locally of almost finite type if any of its open affine subschemes is of almost finite type.

We shall denote the corresponding full subcategories of

$$\mathrm{DGSch}^{\mathrm{aff}} \subset \mathrm{DGSch}_{\mathrm{qs\text{-}qc}} \subset \mathrm{DGSch}$$

by

$$\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}} \subset \mathrm{DGSch}_{\mathrm{aft}} \subset \mathrm{DGSch}_{\mathrm{laft}},$$

respectively.

Definition 3.1.2. *An algebraic stack $\mathcal{Y}$ is* locally of almost finite type *if it admits an atlas $(Z, f : Z \to \mathcal{Y})$, where the DG scheme Z is locally almost of finite type (in which case, for any atlas, the DG scheme Z will have this property).*

3.1.3. We shall now proceed to the definition of prestacks locally almost of finite type. As we mentioned above, the reader is welcome to skip the remainder of this subsection and replace every occurence of the word "prestack" by "algebraic stack". The material here is taken from [GL:Stacks, Sect. 1.3].

First, we fix an integer n, anc consider the full subcategory

$${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \subset \mathrm{DGSch}^{\mathrm{aff}}$$

of n-coconnective affine DG schemes, i.e., those $S = \mathrm{Spec}(A)$, for which $H^{-i}(A) = 0$ for $i > n$.

Let ${}^{\leq n}\mathrm{PreStk}$ denote the category of all functors

$$({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-}\mathrm{Grpd}\,.$$

Definition 3.1.4. *An object ${}^{\leq n}\mathrm{PreStk}$ is said to be* locally of finite type *it it sends filtered limits in ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ to colimits in ∞-Grpd.*

Denote by ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ the full subcategory of ${}^{\leq n}\mathrm{PreStk}$ spanned by objects locally of finite type.

Denote

$${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} := {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \cap \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}\,.$$

We note that ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ identifies with the subcategory of cocompact objects in ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$. Therefore, the Yoneda functor

$${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \to {}^{\leq n}\mathrm{PreStk}$$

sends ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$ to ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$.

It is not difficult to show that the image of entire category

$${}^{\leq n}\mathrm{DGSch}_{\mathrm{lft}} := {}^{\leq n}\mathrm{DGSch} \cap \mathrm{DGSch}_{\mathrm{aft}}$$

under the natural functor ${}^{\leq n}\mathrm{DGSch} \to {}^{\leq n}\mathrm{PreStk}$ is contained in ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$.

3.1.5. We can reformulate the condition on an object $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$ to be locally of finite type in any of the following equivalent ways:

(i) $\mathcal{Y}$ is the left Kan extension along the fully faithful embedding ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$.

(ii) The functor

$$({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}} \to ({}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y}}$$

is cofinal.

(iii) For every $S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ and $y : S \to \mathcal{Y}$, the category of its factorizations as $S \to S' \to \mathcal{Y}$, where $S' \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{ft}}$, is contractible (in particular, non-empty).

3.1.6. We now recall the following definition from [GL:Stacks, Sect. 1.2]:

Definition 3.1.7. *An object* $\mathcal{Y} \in \mathrm{PreStk}$ *is* convergent *if for every* $S \in \mathrm{DGSch}$*, the natural map*

$$\underset{\underset{n}{\longleftarrow}}{lim}\, \mathcal{Y}({}^{\leq n}S) \to \mathcal{Y}(S)$$

is an isomorphism in ∞-Grpd.

In the above formula, the operation $S \mapsto {}^{\leq n}S$ is that of n-coconnective truncation, i.e., if $S = \mathrm{Spec}(A)$, then ${}^{\leq n}S = \mathrm{Spec}(\tau^{\geq -n}(A))$.

For example, all algebraic stacks are convergent, see [GL:Stacks, Proposition 4.5.2].

3.1.8. Finally, we can give the following definition:

Definition 3.1.9. *An object* $\mathcal{Y} \in \mathrm{PreStk}$ *is locally almost of finite type if:*

- *It is convergent;*
- *For every* n*, the restriction* $\mathcal{Y}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}} \in {}^{\leq n}\mathrm{PreStk}$ *belongs to* ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$.

The full subcategory of PreStk spanned by prestacks locally almost of finite type is denoted $\mathrm{PreStk}_{\mathrm{laft}}$.

It is shown in [GL:Stacks, Proposition 4.9.2] that an algebraic stack is locally almost of finite type in the sense of Definition 3.1.2 if and only if it is locally almost of finite type as a prestack in the sense of Definition 3.1.9.

3.1.10. Here is an alternative way to introduce the category $\mathrm{PreStk}_{\mathrm{laft}}$. Let ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$ denote the full subcategory of $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$ spanned by eventually coconnective affine DG schemes.

We have the following assertion (see [GL:Stacks, Sect. 1.3.11]):

Lemma 3.1.11. *The restriction functor under* ${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$ *defines an equivalence*

$$\mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{Funct}\left(({}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}}, \infty\text{-Grpd}\right).$$

The inverse functor is the composition of the left Kan extension along

$${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}} \hookrightarrow {}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}},$$

followed by the right Kan extension along

$${}^{<\infty}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}.$$

Change of conventions: From now and until Sect. 11, all DG schemes, algebraic stacks and prestacks will be assumed locally almost of finite type, unless explicitly specified otherwise.

3.2. **The category** IndCoh**.** For the reader's convenience we shall now summarize some of the key properties of the category IndCoh that will be used in the paper. The general reference for this material in [IndCoh].

3.2.1. Given a quasi-compact DG scheme Z, one introduces the category $\mathrm{IndCoh}(Z)$ as the ind-completion of the category $\mathrm{Coh}(Z)$, the latter being the full subcategory of $\mathrm{QCoh}(Z)$ that consists of bounded complexes with coherent cohomology sheaves; see [IndCoh, Sect. 1.1]. See Sect. 0.6.7 where the notion of ind-completion of a DG category is recalled.

The category $\mathrm{IndCoh}(Z)$ is naturally a module over $\mathrm{QCoh}(Z)$, when the latter is regarded as a monoidal category with respect to the usual tensor product operation, see [IndCoh, Sect. 1.4].

For a morphism $f : Z_1 \to Z_2$ of quasi-compact DG schemes, we have a canonically defined functor

$$f^! : \mathrm{IndCoh}(Z_2) \to \mathrm{IndCoh}(Z_1),$$

see [IndCoh, Corollary 5.2.4].

Moreover, this functor has a canonically defined structure of map between module categories over $\mathrm{QCoh}(Z_2)$, where $\mathrm{QCoh}(Z_2)$ acts on $\mathrm{IndCoh}(Z_1)$ via the monoidal functor $f^* : \mathrm{QCoh}(Z_2) \to \mathrm{QCoh}(Z_1)$; see [IndCoh, Theorem 5.5.5].

The assignment $Z \mapsto \mathrm{IndCoh}(Z)$ with the above !-pullback operation is a functor

$$(\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

denoted $\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}}$, see [IndCoh, Sect. 5.6.1].

We shall denote by ω_Z the object of $\mathrm{IndCoh}(Z)$ equal to $p_Z^!(k)$, where

$$p_Z : Z \to \mathrm{pt}\,.$$

We refer to ω_Z as the "dualizing sheaf" on Z.

The functor $\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}}$ satisfies Zariski descent (see [IndCoh, Proposition 4.2.1].

In fact, something stronger is true: according to [IndCoh, Theorem 8.3.2], the functor $\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}}$ satisfies fppf descent.

The following property of the !-pullback functor will be used in the sequel (see [IndCoh, Proposition 8.1.2]):

Lemma 3.2.2. *Let a morphism $f : Z_1 \to Z_2$ be surjective at the level of geometric points. Then the functor $f^! : \mathrm{IndCoh}(Z_2) \to \mathrm{IndCoh}(Z_1)$ is conservative.*

3.2.3. For two quasi-compact DG schemes Z_1 and Z_2 there is a naturally defined functor

$$\mathrm{IndCoh}(Z_1) \otimes \mathrm{IndCoh}(Z_2) \xrightarrow{\boxtimes} \mathrm{IndCoh}(Z_1 \times Z_2),$$

which is an equivalence by [IndCoh, Proposition 4.6.2]. (The last assertion uses the assumption that $\mathrm{char}(k) = 0$ in an essential way.)

In particular, we obtain a functor

$$\mathrm{IndCoh}(Z) \otimes \mathrm{IndCoh}(Z) \xrightarrow{\boxtimes} \mathrm{IndCoh}(Z \times Z) \xrightarrow{\Delta_Z^!} \mathrm{IndCoh}(Z),$$

that we shall denote by $\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2$. This functor makes $\mathrm{IndCoh}(Z)$ into a symmetric monoidal category with the unit given by ω_Z.

3.2.4. The categories IndCoh(Z) and QCoh(Z) are closely related:

The category IndCoh(Z) has a naturally defined t-structure (induced by one on Coh(Z)). We also have a naturally defined t-exact continuous functor

$$\Psi_Z : \mathrm{IndCoh}(Z) \to \mathrm{QCoh}(Z),$$

characterized by the property that it is the identity functor from $\mathrm{Coh}(Z) \subset \mathrm{IndCoh}(Z)$ to $\mathrm{Coh}(Z) \subset \mathrm{QCoh}(Z)$, see [IndCoh, Secst. 1.1.5 and 1.2.1].

The induced functor on the corresponding eventually coconnective (a.k.a. bounded below) subcategories

$$\mathrm{IndCoh}(Z)^+ \to \mathrm{QCoh}(Z)^+$$

is an equivalence, see [IndCoh, Proposition 1.2.4].

We should add that the t-structure on IndCoh(Z) is compatible with filtered colimits, but it is *not* left-complete, unless Z is a smooth classical scheme, in which case Ψ_Z is an equivalence. In fact, QCoh(Z) is always equivalent to the *left completion* of IndCoh(Z) with respect to its t-structure, [IndCoh, Proposition 1.3.4].

When Z is eventually coconnective, the functor Ψ_Z is a colocalization (see [IndCoh, Proposition 1.5.3]); in particular, in this case it is essentially surjective.

3.2.5. Let $f : Z_1 \to Z_2$ be again a map between quasi-compact DG schemes. There exists a continuous functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z_1) \to \mathrm{IndCoh}(Z_2),$$

uniquely defined by the condition that the diagram

$$\begin{tikzcd} \mathrm{IndCoh}(Z_1) \arrow[r, "\Psi_{Z_1}"] \arrow[d, "f_*^{\mathrm{IndCoh}}"'] & \mathrm{QCoh}(Z_1) \arrow[d, "f_*"] \\ \mathrm{IndCoh}(Z_2) \arrow[r, "\Psi_{Z_1}"] & \mathrm{QCoh}(Z_2). \end{tikzcd} \tag{3.1}$$

commutes, see [IndCoh, Proposition 3.1.1].

The functors of !-pullback and (IndCoh, $*$)-pushforward are endowed with base change isomorphisms for Cartesian squares of DG schemes. I.e., for a Cartesian square

$$\begin{tikzcd} Z_1' \arrow[r, "g_1"] \arrow[d, "f'"'] & Z_1 \arrow[d, "f"] \\ Z_2' \arrow[r, "g_2"] & Z_2 \end{tikzcd} \tag{3.2}$$

there is a canonical isomorphism

$$g_2^! \circ f_*^{\mathrm{IndCoh}} \simeq (f')_*^{\mathrm{IndCoh}} \circ g_1^!; \tag{3.3}$$

see [IndCoh, Theorem 5.2.2] for a precise formulation. Note that in (3.3) there is no adjunction that would produce a morphism in either direction.

For $\mathcal{F}_i \in \mathrm{IndCoh}(Z_i)$, consider the object $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in \mathrm{IndCoh}(Z_1 \times Z_2)$. Applying (3.3) to

$$\begin{tikzcd} Z_1 \arrow[r, "\mathrm{Graph}_f"] \arrow[d, "f"'] & Z_1 \times Z_2 \arrow[d, "f \times \mathrm{id}"] \\ Z_2 \arrow[r, "\Delta_{Z_2}"] & Z_2 \times Z_2, \end{tikzcd}$$

we deduce that f satisfies the projection formula for IndCoh:

$$\mathcal{F}_2 \overset{!}{\otimes} f_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \simeq f_*^{\mathrm{IndCoh}}(f^!(\mathcal{F}_2) \overset{!}{\otimes} \mathcal{F}_1). \tag{3.4}$$

3.2.6. Assume that the map f is eventually coconnective; see [IndCoh, Definition 3.5.2], where this notion is introduced. Note that this is equivalent to f being finite Tor-dimension, see [IndCoh, Lemma 3.6.3].

In this case there also exists a functor

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(Z_2) \to \mathrm{IndCoh}(Z_1),$$

uniquely defined by the condition that the diagram

$$\begin{tikzcd} \mathrm{IndCoh}(Z_1) \arrow[r, "\Psi_{Z_1}"] & \mathrm{QCoh}(Z_1) \\ \mathrm{IndCoh}(Z_2) \arrow[u, "f^{\mathrm{IndCoh},*}"] \arrow[r, "\Psi_{Z_1}"] & \mathrm{QCoh}(Z_2). \arrow[u, "f^*"'] \end{tikzcd} \tag{3.5}$$

commutes, see [IndCoh, Proposition 3.5.4], and which is the left adjoint to f_*^{IndCoh}.

For a Cartesian diagram (3.2), in which the vertical arrows are eventually coconnective, the natural transformation

$$(f')^{\mathrm{IndCoh},*} \circ g_2^! \to g_1^! \circ g^{\mathrm{IndCoh},*} \tag{3.6}$$

that arises by adjunction from (3.3), is an isomorphism (see [IndCoh, Proposition 7.1.6]).

If the map f is smooth (or, more generally, Gorenstein), then we have:

$$f^!(-) \simeq \mathcal{K}_{Z_1/Z_2} \otimes f^{\mathrm{IndCoh},*}(-), \tag{3.7}$$

where $\mathcal{K}_{Z_1/Z_2}$ is the relative dualizing graded line bundle (see [IndCoh, Proposition 7.3.8]). In the above formula, tensor product is understood in the sense of the monoidal action of $\mathrm{QCoh}(Z)$ on $\mathrm{IndCoh}(Z)$.

For a Cartesian diagram (3.2) with the horizontal maps being eventually coconnective, the natural transformation

$$g_2^{\mathrm{IndCoh},*} \circ f_*^{\mathrm{IndCoh}} \to (f')^{\mathrm{IndCoh},*} \circ g_1^{\mathrm{IndCoh},*}, \tag{3.8}$$

obtained by adjunction from

$$f_*^{\mathrm{IndCoh}} \circ (g_2)_*^{\mathrm{IndCoh}} \simeq (g_1)_*^{\mathrm{IndCoh}} \circ (f')_*^{\mathrm{IndCoh}},$$

is an isomorphism, see [IndCoh, Lemma 3.6.9].

3.2.7. Let now $\mathcal{Y}$ be a prestack. We define the category $\mathrm{IndCoh}(\mathcal{Y})$ as

$$\underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}}{\underleftarrow{lim}} \mathrm{IndCoh}(S), \tag{3.9}$$

where we view the assignment $(S, g) \rightsquigarrow \mathrm{IndCoh}(S)$ as a functor between ∞-categories

$$((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

obtained by restriction under the forgetful map $(\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}} \to \mathrm{DGSch}_{\mathrm{aft}}$ of the functor

$$\mathrm{IndCoh}^!_{\mathrm{DGSch\,aft}} : \mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

mentioned above. As in the case of QCoh, the limit is taken in the $(\infty, 1)$-category $\mathrm{DGCat}_{\mathrm{cont}}$.

Concretely, an object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y})$ is an assignment for

$$(g : S \to \mathcal{Y}) \in (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}} \rightsquigarrow g^!(\mathcal{F}) \in \mathrm{IndCoh}(S),$$

and of a homotopy-coherent system of isomorphisms

$$f^!(g^!(\mathcal{F})) \simeq (g \circ f)^!(\mathcal{F}) \in \mathrm{IndCoh}(S')$$

for $f : S' \to S$.

In forming the above limit we can replace the category $\mathrm{DGSch}_{\mathrm{aft}}$ of quasi-compact DG schemes by $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$ of affine DG schemes; this is due to the Zariski descent property of IndCoh, see [IndCoh, Corollaries 10.2.2 and 10.5.5]. Furthermore, we can replace the category $\mathrm{DGSch}_{\mathrm{aft}}$ (resp., $\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$) by any of the indexing categories A that appear in Sect. 1.2.5.

The compatibility of !-pullbacks with the action of QCoh implies that the category $\mathrm{IndCoh}(\mathcal{Y})$ has a natural structure of module over the monoidal category $\mathrm{QCoh}(\mathcal{Y})$.

3.2.8. If $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a map of prestacks, we have a tautologically defined functor $\pi^! : \mathcal{Y}_1 \to \mathcal{Y}_2$.

In particular, for any $\mathcal{Y}$, we obtain a canonical object $\omega_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y})$ equal to $p^!_{\mathcal{Y}}(k)$, where $p_{\mathcal{Y}} : \mathcal{Y} \to \mathrm{pt}$. We refer to $\omega_{\mathcal{Y}}$ as "the dualizing sheaf" on $\mathcal{Y}$.

For two prestacks $\mathcal{Y}_1$ and $\mathcal{Y}_2$ there exists a naturally defined functor

$$\mathrm{IndCoh}(\mathcal{Y}_1) \otimes \mathrm{IndCoh}(\mathcal{Y}_1) \xrightarrow{\boxtimes} \mathrm{IndCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2).$$

In particular, as in the case of schemes, $\mathrm{IndCoh}(\mathcal{Y})$ acquires a structure of symmetric monoidal category via the operation $\overset{!}{\otimes}$.

3.2.9. Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a schematic and quasi-compact map between prestacks. Then the functor of direct image on IndCoh for DG schemes gives rise to a functor

$$\pi^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(\mathcal{Y}_1) \to \mathrm{IndCoh}(\mathcal{Y}_2).$$

Namely, for $(S_2, g_2) \in (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}}$, we set

$$g^!_2(\pi^{\mathrm{IndCoh}}_*(-)) := (\pi_S)^{\mathrm{IndCoh}}_* \circ g^!_1(-)$$

for the morphisms in the Cartesian diagram

$$\begin{tikzcd} S_1 \arrow[r, "g_1"] \arrow[d, "\pi_S"'] & \mathcal{Y}_1 \arrow[d, "\pi"] \\ S_2 \arrow[r, "g_2"] & \mathcal{Y}_2. \end{tikzcd}$$

The data of compatibility of the assignment

$$(S_2, g_2) \rightsquigarrow (\pi_S)^{\mathrm{IndCoh}}_* \circ g^!_1(-)$$

under !-pullbacks for maps in $(\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}}$ is given by base change isomorphisms (3.3); see [IndCoh, Sect. 10.6].

The resulting functor π^{IndCoh}_* is itself also endowed with base change isomorphisms with respect to !-pullbacks for Cartesian diagrams of prestacks

$$\begin{tikzcd} \mathcal{Y}'_1 \arrow[r, "\phi_1"] \arrow[d, "\pi'"'] & \mathcal{Y}_1 \arrow[d, "\pi"] \\ \mathcal{Y}'_2 \arrow[r, "\phi_2"] & \mathcal{Y}_2 \end{tikzcd} \tag{3.10}$$

where the vertical maps are schematic and quasi-compact.

By construction, the projection formula for maps between quasi-compact schemes, i.e., (3.4), implies one for π. That is, we have a functorial isomorphism

$$\mathcal{F}_2 \overset{!}{\otimes} \pi_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \simeq \pi_*^{\mathrm{IndCoh}}(\pi^!(\mathcal{F}_2) \overset{!}{\otimes} \mathcal{F}_1), \quad \mathcal{F}_i \in \mathrm{IndCoh}(\mathcal{Y}_i).$$

3.2.10. Let $\mathcal{Y}_i$ be prestacks, and let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism which is k-representable for some k. In this paper we will only need the cases of either π being schematic, or 1-representable (the latter means that the base change of π by an affine DG scheme yields a 1-Artin stack).

Assume also that π is eventually coconnective, see [IndCoh, Sect. 11.1.2]. In this case, by [IndCoh, Sect. 11.6], we have a continuous functor

$$\pi^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Y}_2) \to \mathrm{IndCoh}(\mathcal{Y}_1).$$

For a Cartesian diagram (3.10), in which the vertical arrows are k-representable and eventually coconnective, we have a canonical isomorphism

$$(\pi')^{\mathrm{IndCoh},*} \circ \phi_2^! \simeq \phi_1^! \circ \pi^{\mathrm{IndCoh},*}, \tag{3.11}$$

see [IndCoh, Proposition 11.6.2]. Note that unlike (3.6), in (3.11) there is no a priori map in either direction.

If f is smooth (or, more generally, Gorenstein), the functors $\pi^{\mathrm{IndCoh},*}$ and $\pi^!$ are related by the formula

$$\pi^!(-) \simeq \mathcal{K}_{\mathcal{Y}_1/\mathcal{Y}_2} \otimes \pi^{\mathrm{IndCoh},*}(-), \tag{3.12}$$

where $\mathcal{K}_{\mathcal{Y}_1/\mathcal{Y}_2}$ is the relative dualizing line bundle. This is not explicitly stated in [IndCoh], but can be obtained by combining the functorial isomorphisms (3.7) for morphisms between DG schemes, and (3.11).

If π is schematic and quasi-compact, the functors $(\pi^{\mathrm{IndCoh},*}, \pi_*^{\mathrm{IndCoh}})$ form an adjoint pair. The latter fact is not stated explicitly in [IndCoh] either, but follows from (3.11) via an analog of Sect. 1.2.5(ii) for IndCoh.

3.2.11. When $\mathcal{Y}$ is an algebraic stack, the category $\mathrm{IndCoh}(\mathcal{Y})$ can be described more explicitly.

First, as in Sect. 1.2.5(iv), in the formation of the limit (3.9), we can replace the category $(\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}}$ by $\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}$, see [IndCoh, Corollary 11.2.4].

Furthermore, when we use $(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}$ as the indexing category, $\mathrm{IndCoh}(\mathcal{Y})$ can be also realized as the limit

$$\underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \mathrm{IndCoh}(S), \tag{3.13}$$

where now for a morphism $f : S' \to S'$ in $\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}$, the transition functor $\mathrm{IndCoh}(S) \to \mathrm{IndCoh}(S')$ is $f^{\mathrm{IndCoh},*}$, see [IndCoh, Sect. 11.3 and Proposition 11.4.3].

If $f : Z \to \mathcal{Y}$ is a smooth atlas, the naturally defined functor

$$\mathrm{IndCoh}(Z) \to \mathrm{Tot}(\mathrm{IndCoh}(Z^\bullet/\mathcal{Y})) \tag{3.14}$$

is an equivalence. In the above formula, the cosimplicial category $\mathrm{IndCoh}(Z^\bullet/\mathcal{Y})$ is formed by using either the !-pullback or $(\mathrm{IndCoh}, *)$-pullback functors along the simplicial DG scheme $Z^\bullet/\mathcal{Y}$. See [IndCoh, Corollary 11.3.4] for the proof.

For a Cartesian diagram (3.10) consisting of algebraic stacks, in which the vertical arrows are schematic and quasi-compact and the horizontal ones are eventually coconnective, we have a canonical isomorphism

$$\phi_2^{\mathrm{IndCoh},*} \circ \pi_*^{\mathrm{IndCoh}} \simeq (\pi')_*^{\mathrm{IndCoh}} \circ \phi_1^{\mathrm{IndCoh},*}. \tag{3.15}$$

It is obtained from the natural transformation (3.8) using (3.13). Note again that unless the vertical arrows are also eventually coconnective or the horizontal maps schematic and quasi-compact, there is a priori no morphism in either direction in (3.15).

3.2.12. For $\mathcal{Y}$ an algebraic stack, the category $\mathrm{IndCoh}(\mathcal{Y})$ has a t-structure and the functor

$$\Psi_{\mathcal{Y}} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})$$

with the same properties as those for schemes, reviewed in Sect. 3.2.4 above, see [IndCoh, Sect. 11.7.1 and Proposition 11.7.5]. Namely, the functor $\Psi_{\mathcal{Y}}$ is determined uniquely by the requirement that for $(S, g) \in \mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}$, the diagram

$$\begin{CD}
\mathrm{IndCoh}(\mathcal{Y}) @>{g^{\mathrm{IndCoh},*}}>> \mathrm{IndCoh}(S) \\
@V{\Psi_{\mathcal{Y}}}VV @VV{\Psi_S}V \\
\mathrm{QCoh}(\mathcal{Y}) @>{g^*}>> \mathrm{QCoh}(S)
\end{CD}$$

is supplied with a commutativity isomorphism, functorially in (S, g). The t-structure on $\mathrm{IndCoh}(\mathcal{Y})$ is determined by the condition that the functors $g^{\mathrm{IndCoh},*}$ be t-exact.

If $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ is an eventually coconnective morphism between algebraic stacks, we have a commutative diagram

$$\begin{CD}
\mathrm{IndCoh}(\mathcal{Y}_1) @>{\Psi_{\mathcal{Y}_1}}>> \mathrm{QCoh}(\mathcal{Y}_1) \\
@A{\pi^{\mathrm{IndCoh},*}}AA @AA{\pi^*}A \\
\mathrm{IndCoh}(\mathcal{Y}_2) @>{\Psi_{\mathcal{Y}_2}}>> \mathrm{QCoh}(\mathcal{Y}_2).
\end{CD} \tag{3.16}$$

For a schematic and quasi-compact map $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ between algebraic stacks, we have a commutative diagram

$$\begin{CD}
\mathrm{IndCoh}(\mathcal{Y}_1) @>{\Psi_{\mathcal{Y}_1}}>> \mathrm{QCoh}(\mathcal{Y}_1) \\
@V{\pi_*^{\mathrm{IndCoh}}}VV @VV{\pi_*}V \\
\mathrm{IndCoh}(\mathcal{Y}_2) @>{\Psi_{\mathcal{Y}_2}}>> \mathrm{QCoh}(\mathcal{Y}_2).
\end{CD} \tag{3.17}$$

It arises from the corresponding commutative diagrams in the case of DG schemes , i.e., (3.1), using the functorial isomorphisms (3.15).

3.2.13. For an algebraic stack $\mathcal{Y}$, we shall denote by $\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, -) : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Vect}$ the *not necessarily* continuous functor equal to

$$\Gamma(\mathcal{Y}, -) \circ \Psi_{\mathcal{Y}}.$$

From Lemma 1.3.13 we obtain that for $\mathcal{F} \in \operatorname{IndCoh}(\mathcal{Y})$ there is a canonical isomorphism

$$\Gamma^{\operatorname{IndCoh}}(\mathcal{Y}, \mathcal{F}) \simeq \underset{(S,g)\in(\operatorname{DGSch}_{/\mathcal{Y},\operatorname{smooth}})^{\operatorname{op}}}{\underset{\longleftarrow}{lim}} \Gamma\left(S, g^*(\Psi_{\mathcal{Y}}(\mathcal{F}))\right) \simeq \simeq \underset{(S,g)\in(\operatorname{DGSch}_{/\mathcal{Y},\operatorname{smooth}})^{\operatorname{op}}}{\underset{\longleftarrow}{lim}} \Gamma^{\operatorname{IndCoh}}(S, g^{\operatorname{IndCoh},*}(\mathcal{F})). \tag{3.18}$$

3.3. **The coherent subcategory.** Let $\mathcal{Y}$ be an algebraic stack.

3.3.1. We define $\operatorname{Coh}_{\operatorname{Ind}}(\mathcal{Y})$ to be the full subcategory of $\operatorname{IndCoh}(\mathcal{Y})$ consisting of those objects $\mathcal{F}$, for which for any affine DG scheme S equipped with a smooth map $g : S \to \mathcal{Y}$, the corresponding object $g^{\operatorname{IndCoh},*}(\mathcal{F})$ belongs to $\operatorname{Coh}(S) \subset \operatorname{IndCoh}(S)$. This condition is enough to check for any fixed collection (S_α, g_α) such that the map $\underset{\alpha}{\sqcup} S_\alpha \to \mathcal{Y}$ is surjective.

Note that in the above definition, we can replace the functors $g^{\operatorname{IndCoh},*}$ by $g^!$. This follows from either (3.11) or (3.12).

We define $\operatorname{Coh}_{\operatorname{Q}}(\mathcal{Y})$ to be the full subcategory of $\operatorname{QCoh}(\mathcal{Y})$ consisting of those objects $\mathcal{F}$, for which for any affine DG scheme S equipped with a smooth map $g : S \to \mathcal{Y}$, the corresponding object $g^*(\mathcal{F})$ belongs to $\operatorname{Coh}(S) \subset \operatorname{QCoh}(S)$. This condition is enough to check for any fixed collection (S_α, g_α) such that the map $\underset{\alpha}{\sqcup} S_\alpha \to \mathcal{Y}$ is surjective.

We claim:

Lemma 3.3.2. *The functor $\Psi_{\mathcal{Y}}$ defines an equivalence $\operatorname{Coh}_{\operatorname{Ind}}(\mathcal{Y}) \to \operatorname{Coh}_{\operatorname{Q}}(\mathcal{Y})$.*

Proof. Follows by combining (3.16) with (3.13) and Sect. 1.2.5(iv). □

From now on, we will identify $\operatorname{Coh}_{\operatorname{Ind}}(\mathcal{Y})$ with $\operatorname{Coh}_{\operatorname{Q}}(\mathcal{Y})$ and denote the resulting category simply by $\operatorname{Coh}(\mathcal{Y})$, unless a confusion is likely to occur.

3.3.3. Consider the ind-completion $\operatorname{Ind}(\operatorname{Coh}(\mathcal{Y}))$ of the category $\operatorname{Coh}(\mathcal{Y})$ (see Sect. 0.6.7 where the notion of ind-completion of a DG category is recalled). One has a tautologically defined continuous functor

$$\operatorname{Ind}(\operatorname{Coh}(\mathcal{Y})) \to \operatorname{Ind}\operatorname{Coh}(\mathcal{Y}). \tag{3.19}$$

However, it is not true that this functor is always an equivalence. For example, it is typically not an equivalence for non quasi-compact schemes.

3.3.4. The main result of this section is the following theorem, which says that $\operatorname{IndCoh}(\mathcal{Y}) = \operatorname{Ind}(\operatorname{Coh}(\mathcal{Y}))$ if $\mathcal{Y}$ is QCA (see Definition 1.1.8).

Theorem 3.3.5. *Assume that a stack $\mathcal{Y}$ is QCA. Then the category $\operatorname{IndCoh}(\mathcal{Y})$ is compactly generated. Moreover, its subcategory of compact objects equals $\operatorname{Coh}(\mathcal{Y})$.*

3.3.6. The proof will be given in Sects. 3.4-3.5 (it is based on Theorem 1.4.2). This theorem will imply a number of favorable properties of the category IndCoh; these will be established in Sect. 4, see Sects. 4.2 and 4.3).

3.4. **Description of compact objects of** $\operatorname{IndCoh}(\mathcal{Y})$**.**

3.4.1. First, we claim:

Proposition 3.4.2.
(a) *For any algebraic stack, the subcategory* $\mathrm{IndCoh}(\mathcal{Y})^c \subset \mathrm{IndCoh}(\mathcal{Y})$ *is contained in* $\mathrm{Coh}(\mathcal{Y})$.
(b) *If* $\mathcal{Y}$ *is QCA then* $\mathrm{IndCoh}(\mathcal{Y})^c = \mathrm{Coh}(\mathcal{Y})$.

Proof of point (a). When need to show that for any affine DG scheme S equipped with a smooth map $g : S \to \mathcal{Y}$, the functor $g^{\mathrm{IndCoh},*}$ sends $\mathrm{IndCoh}(\mathcal{Y})^c$ to $\mathrm{IndCoh}(S)^c = \mathrm{Coh}(S)$.

Since $\mathcal{Y}$ is an algebraic stack, the morphism g is schematic and quasi-compact. Hence, the functor $g^{\mathrm{IndCoh},*}$ admits a continuous right adjoint, namely, g_*^{IndCoh} (see Sect. 3.2.10). This implies the required assertion.

□

Remark 3.4.3. For point (b), we need to show that, when $\mathcal{Y}$ is QCA and $\mathcal{F} \in \mathrm{Coh}(\mathcal{Y})$, the assignment

$$\mathcal{F}' \mapsto \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}')$$

commutes with colimits in $\mathcal{F}'$. The idea of the proof is that to $\mathcal{F}$ and $\mathcal{F}'$ one can assign their *internal* Hom object

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}') \in \mathrm{QCoh}(\mathcal{Y}),$$

whose formation commutes with colimits in $\mathcal{F}'$, and such that

$$\mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}') \simeq \Gamma\left(\mathcal{Y}, \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}')\right).$$

Then the assertion of point (b) of the proposition would follow from Theorem 1.4.2.

Proof of point (b). Let $\mathcal{Y}$ be QCA and $\mathcal{F} \in \mathrm{Coh}(\mathcal{Y})$ and $\mathcal{F}' \in \mathrm{IndCoh}(\mathcal{Y})$. We have:

$$\mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}') \simeq \underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \mathcal{M}aps_{\mathrm{IndCoh}(S)}(g^{\mathrm{IndCoh},*}(\mathcal{F}), g^{\mathrm{IndCoh},*}(\mathcal{F}')).$$

For every $(S, g) \in \mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{smooth}}$ consider the object

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(g^{\mathrm{IndCoh},*}(\mathcal{F}), g^{\mathrm{IndCoh},*}(\mathcal{F}')) \in \mathrm{QCoh}(S),$$

(see [GL:DG], Sect. 5.1.). Namely, for $\mathcal{E} \in \mathrm{QCoh}(S)$,

$$\mathcal{M}aps_{\mathrm{QCoh}(S)}(\mathcal{E}, \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(g^{\mathrm{IndCoh},*}(\mathcal{F}), g^{\mathrm{IndCoh},*}(\mathcal{F}'))) \simeq$$
$$\simeq \mathcal{M}aps_{\mathrm{IndCoh}(S)}(\mathcal{E} \otimes g^{\mathrm{IndCoh},*}(\mathcal{F}), g^{\mathrm{IndCoh},*}(\mathcal{F}')),$$

where $- \otimes -$ denotes the action of $\mathrm{QCoh}(S)$ on $\mathrm{IndCoh}(S)$.

Since $g^{\mathrm{IndCoh},*}(\mathcal{F}) \in \mathrm{Coh}(S) = \mathrm{IndCoh}(S)^c$, and $\mathrm{QCoh}(S)$ is compactly generated, by [GL:DG, Lemma 5.1.1], the assignment

$$\mathcal{F}' \mapsto \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(g^{\mathrm{IndCoh},*}(\mathcal{F}), g^{\mathrm{IndCoh},*}(\mathcal{F}'))$$

commutes with colimits.

By construction, for every map $f : \widetilde{S} \to S$ in $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{smooth}}$, there is a canonical map

$$f^*(\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(g^{\mathrm{IndCoh},*}(\mathcal{F}), g^{\mathrm{IndCoh},*}(\mathcal{F}'))) \to \tag{3.20}$$
$$\to \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\widetilde{S})}(\widetilde{g}^{\mathrm{IndCoh},*}(\mathcal{F}), \widetilde{g}^{\mathrm{IndCoh},*}(\mathcal{F}')),$$

where $\widetilde{g} = g \circ f$.

Lemma 3.4.4. *The map* (3.20) *is an isomorphism.*

The proof will be given in Sect. 3.4.6. Thus, we obtain that the assignment

$$(S,g) \mapsto \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(g^{\mathrm{IndCoh},*}(\mathcal{F}), g^{\mathrm{IndCoh},*}(\mathcal{F}'))$$

defines an object

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},\mathcal{F}') \in \mathrm{QCoh}(\mathcal{Y}).$$

Moreover, the functor

$$\mathcal{F}' \mapsto \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},\mathcal{F}')$$

commutes with colimits.

By construction,

$$\mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{Y})}(\mathcal{F},\mathcal{F}') \simeq \Gamma\left(\mathcal{Y}, \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},\mathcal{F}')\right). \tag{3.21}$$

Now, the required assertion follows from Theorem 1.4.2.

□

Remark 3.4.5. It is easy to see that the object $\underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},\mathcal{F}')$ introduced above is the internal Hom of $\mathcal{F}$ and $\mathcal{F}'$ in the sense of [GL:DG, Sect. 5.1], i.e., for $\mathcal{E} \in \mathrm{QCoh}(\mathcal{Y})$, we have

$$\mathcal{M}aps(\mathcal{E}, \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},\mathcal{F}')) \simeq \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{Y})}(\mathcal{E}\otimes\mathcal{F},\mathcal{F}').$$

3.4.6. *Proof of Lemma 3.4.4.* Let $f : \widetilde{S} \to S$ be an eventually coconnective map of affine DG schemes, and $\mathcal{F} \in \mathrm{Coh}(S)$, and $\mathcal{F}' \in \mathrm{IndCoh}(S)$. We claim that the natural map

$$f^*(\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F},\mathcal{F}')) \to \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\widetilde{S})}(f^{\mathrm{IndCoh},*}(\mathcal{F}), f^{\mathrm{IndCoh},*}(\mathcal{F}')))$$

is an isomorphism. The latter is equivalent to

$$f_*(\mathcal{O}_{\widetilde{S}}) \underset{\mathcal{O}_S}{\otimes} \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F},\mathcal{F}') \to f_*(\underline{\mathrm{Hom}}_{\mathrm{QCoh}(\widetilde{S})}(f^{\mathrm{IndCoh},*}(\mathcal{F}), f^{\mathrm{IndCoh},*}(\mathcal{F}')))$$

being an isomorphism at the level of global sections.

Now, since $\mathcal{F}$ is a compact object of $\mathrm{IndCoh}(S)$, by [GL:DG, Lemma 5.1.1], we have:

$$f_*(\mathcal{O}_{\widetilde{S}}) \underset{\mathcal{O}_S}{\otimes} \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F},\mathcal{F}') \simeq \underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}, f_*(\mathcal{O}_{\widetilde{S}}) \underset{\mathcal{O}_S}{\otimes} \mathcal{F}').$$

Thus, we need to show that

$$\mathcal{M}aps_{\mathrm{IndCoh}(S)}(\mathcal{F}, f_*(\mathcal{O}_{\widetilde{S}}) \underset{\mathcal{O}_S}{\otimes} \mathcal{F}') \to \mathcal{M}aps_{\mathrm{IndCoh}(\widetilde{S})}(f^{\mathrm{IndCoh},*}(\mathcal{F}), f^{\mathrm{IndCoh},*}(\mathcal{F}')) \simeq$$
$$\simeq \mathcal{M}aps_{\mathrm{IndCoh}(S)}(\mathcal{F}, f_*^{\mathrm{IndCoh}} \circ f^{\mathrm{IndCoh},*}(\mathcal{F}')).$$

I.e., it is sufficient to prove that the map

$$f_*(\mathcal{O}_{\widetilde{S}}) \underset{\mathcal{O}_S}{\otimes} \mathcal{F}' \to f_*^{\mathrm{IndCoh}} \circ f^{\mathrm{IndCoh},*}(\mathcal{F}')$$

is an isomorphism. However, the latter is the content of [IndCoh, Proposition 3.6.11].

□

3.5. **The category** $\mathrm{Coh}(\mathcal{Y})$ **generates** $\mathrm{IndCoh}(\mathcal{Y})$**.**

Theorem 3.3.5 follows from Proposition 3.4.2 and the next one.

Proposition 3.5.1. *If* $\mathcal{Y}$ *is QCA then the subcategory* $\mathrm{Coh}(\mathcal{Y})^\heartsuit$ *generates* $\mathrm{IndCoh}(\mathcal{Y})$*.*

The proof of Proposition 3.5.1, given below, is parallel to the proof of Theorem 1.4.10 given in Sect. 2.6.

Remark 3.5.2. In some sense, the proof of Proposition 3.5.1 is simpler because for IndCoh the !-pullback is a continuous functor (unlike the situation of Sect. 2.6.8 and Remark 2.6.9). So one may prefer to deduce Theorem 1.4.10 from Proposition 3.5.1 using the functor $\Psi_{\mathcal{Y}}$: $\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})$, which is essentially surjective if $\mathcal{Y}$ is eventually coconnective.

3.5.3. First, just as in Sect. 2.6, one can assume that $\mathcal{Y}$ is classical and reduced.

Let $\mathcal{Y}_i$ be the locally closed substacks of $\mathcal{Y}$ given by Proposition 2.3.4. With no restriction of generality, we can assume that all $\mathcal{Y}_i$ are smooth. In this case $\mathrm{IndCoh}(\mathcal{Y}_i) \simeq \mathrm{QCoh}(\mathcal{Y}_i)$, so Lemmas 2.6.3 and 2.6.4 imply that $\mathrm{IndCoh}(\mathcal{Y}_i)$ is generated by $\mathrm{Coh}(\mathcal{Y}_i)^\heartsuit$.

Hence, to prove the theorem, it suffices to prove the following analog of Lemma 2.6.5:

Lemma 3.5.4. *Let* $\mathcal{X}$ *and* $\overset{\circ}{\mathcal{Y}}$ *be as in Lemma 2.3.2. Then if the assertion of Proposition 3.5.1 holds for* $\mathcal{X}$ *and* $\overset{\circ}{\mathcal{Y}}$*, then it holds also for* $\mathcal{Y}$*.*

Proof. We have to show that if $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y})$ and

$$\mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{Y})}(\mathcal{E}, \mathcal{F}) = 0 \text{ for all } \mathcal{E} \in \mathrm{Coh}(\mathcal{Y})^\heartsuit$$

then $\mathcal{F} = 0$.

Consider the exact triangle

$$(\mathcal{F})_{\mathcal{X}} \to \mathcal{F} \to \jmath_*^{\mathrm{IndCoh}} \circ \jmath^{\mathrm{IndCoh},*}(\mathcal{F}), \tag{3.22}$$

where

$$(\mathcal{F})_{\mathcal{X}} := \mathrm{Cone}\left(\jmath_*^{\mathrm{IndCoh}} \circ \jmath^{\mathrm{IndCoh},*}(\mathcal{F})\right)[-1].$$

By [IndCoh, Proposition 4.1.7] (which is applicable to algebraic stacks),

$$(\mathcal{F})_{\mathcal{X}} \Leftrightarrow \imath^!(\mathcal{F}) = 0.$$

For any $\mathcal{F}' \in \mathrm{Coh}(\mathcal{X})^\heartsuit$ one has

$$\mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}', \imath^!(\mathcal{F})) = \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{Y})}(\imath_*^{\mathrm{IndCoh}}(\mathcal{F}'), \mathcal{F}) = 0,$$

and $\imath_*^{\mathrm{IndCoh}}(\mathcal{F}') \in \mathrm{Coh}(\mathcal{Y})^\heartsuit$. So, the assumption that Proposition 3.5.1 holds for $\mathcal{X}$ implies that $\imath^!(\mathcal{F}) = 0$. Therefore, $(\mathcal{F})_{\mathcal{X}} = 0$, and, hence,

$$\mathcal{F} \to \jmath_*^{\mathrm{IndCoh}} \circ \jmath^{\mathrm{IndCoh},*}(\mathcal{F})$$

is an isomorphism.

In particular, for every $\mathcal{E} \in \mathrm{IndCoh}(\mathcal{Y})$, we have:

$$\begin{aligned} \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{Y})}(\mathcal{E}, \mathcal{F}) \simeq \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{Y})}\left(\mathcal{E}, \jmath_*^{\mathrm{IndCoh}} \circ \jmath^{\mathrm{IndCoh},*}(\mathcal{F})\right) \simeq \\ \simeq \mathcal{M}aps_{\mathrm{IndCoh}(\overset{\circ}{\mathcal{Y}})}\left(\jmath^{\mathrm{IndCoh},*}(\mathcal{E}), \jmath^{\mathrm{IndCoh},*}(\mathcal{F})\right). \end{aligned} \tag{3.23}$$

Now we use the following lemma, which immediately follows from [LM, Corollary 15.5].

Lemma 3.5.5. *For every* $\overset{\circ}{\mathcal{E}} \in \mathrm{Coh}(\overset{\circ}{\mathcal{Y}})^\heartsuit$*, there exists* $\mathcal{E} \in \mathrm{Coh}(\mathcal{Y})^\heartsuit$ *such that* $\jmath^*(\mathcal{E}) \simeq \overset{\circ}{\mathcal{E}}$*.*

By (3.23), for every $\overset{\circ}{\mathcal{E}} \in \mathrm{Coh}(\overset{\circ}{\mathcal{Y}})^{\heartsuit}$ and the corresponding $\mathcal{E} \in \mathrm{Coh}(\mathcal{Y})^{\heartsuit}$, we have:

$$\mathcal{M}aps_{\mathrm{IndCoh}(\overset{\circ}{\mathcal{Y}})}\left(\overset{\circ}{\mathcal{E}}, \jmath^{\mathrm{IndCoh},*}(\mathcal{F})\right) \simeq \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{Y})}(\mathcal{E}, \mathcal{F}) = 0.$$

Hence, $\jmath^{\mathrm{IndCoh},*}(\mathcal{F}) = 0$, by the assumption that Proposition 3.5.1 holds for $\overset{\circ}{\mathcal{Y}}$.

Thus, we have $(\mathcal{F})_{\mathcal{X}} = 0$ and $\jmath^{\mathrm{IndCoh},*}(\mathcal{F}) = 0$, and by (3.22), this implies that $\mathcal{F} = 0$.

□

3.6. **Direct image functor on** IndCoh**.** As an application of Theorem 3.3.5, we shall now construct a functor π_*^{IndCoh} for a morphism $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ between QCA algebraic stacks. [12]

3.6.1. We claim that in this case there exists a unique continuous functor

$$\pi_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{Y}_1) \to \mathrm{IndCoh}(\mathcal{Y}_2),$$

which is left t-exact and which makes the following diagram commute:

$$\begin{CD}
\mathrm{IndCoh}(\mathcal{Y}_1) @>{\Psi_{\mathcal{Y}_1}}>> \mathrm{QCoh}(\mathcal{Y}_1) \\
@V{\pi_*^{\mathrm{IndCoh}}}VV @VV{\pi_*}V \\
\mathrm{IndCoh}(\mathcal{Y}_2) @>{\Psi_{\mathcal{Y}_2}}>> \mathrm{QCoh}(\mathcal{Y}_2).
\end{CD}$$

Indeed, the functor π_*^{IndCoh} is obtained as the ind-extension of the functor

$$\mathrm{Coh}(\mathcal{Y}_1) \to \mathrm{IndCoh}(\mathcal{Y}_2)$$

equal to the composition

$$\mathrm{Coh}(\mathcal{Y}_1) \hookrightarrow \mathrm{QCoh}(\mathcal{Y}_1)^+ \xrightarrow{\pi_*} \mathrm{QCoh}(\mathcal{Y}_2)^+ \simeq \mathrm{IndCoh}(\mathcal{Y}_2)^+ \hookrightarrow \mathrm{IndCoh}(\mathcal{Y}_2),$$

where $\mathrm{QCoh}(\mathcal{Y}_2)^+ \simeq \mathrm{IndCoh}(\mathcal{Y}_2)^+$ is the equivalence inverse to that induced by $\Psi_{\mathcal{Y}_2}$, see Sect. 3.2.12.

It is easy to see that when π is schematic and quasi-compact, the above functor π_*^{IndCoh} is canonically isomorphic to the one in Sect. 3.2.9. This follows from the defining property of π_*^{IndCoh}, using the commutative diagram (3.17).

3.6.2. Consider the particular case when $\mathcal{Y}_1 = \mathcal{Y}$ and $\mathcal{Y}_2 = \mathrm{pt}$, and $\pi = p_{\mathcal{Y}}$. Recall the functor $\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, -)$, see Sect. 3.2.13.

Since the functor $\Gamma(\mathcal{Y}, -) : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{pt}$ is continuous, so is the functor $\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, -)$.

We obtain that we have a canonical isomorphism of functors

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, -) \simeq (p_{\mathcal{Y}})^{\mathrm{IndCoh}}. \tag{3.24}$$

(Indeed, the two functors tautologically coincide on $\mathrm{Coh}(\mathcal{Y}) \subset \mathrm{IndCoh}(\mathcal{Y})$, and the isomorphism on all of $\mathrm{IndCoh}(\mathcal{Y})$ follows by continuity.)

[12] For this construction to make sense we only need $\mathcal{Y}_1$ to be QCA, while $\mathcal{Y}_2$ may be arbitrary.

3.6.3. The defining property of π_*^{IndCoh} implies that it is compatible with compositions. I.e., if

$$\mathcal{Y}_1 \xrightarrow{\pi} \mathcal{Y}_2 \xrightarrow{\phi} \mathcal{Y}_3$$

are maps between QCA algebraic stacks, we have

$$\phi_*^{\mathrm{IndCoh}} \circ \pi_*^{\mathrm{IndCoh}} \simeq (\phi \circ \pi)_*^{\mathrm{IndCoh}}.$$

This follows from the diagram

$$\begin{CD}
\mathrm{IndCoh}(\mathcal{Y}_1) @>{\Psi_{\mathcal{Y}_1}}>> \mathrm{QCoh}(\mathcal{Y}_1) \\
@V{\pi_*^{\mathrm{IndCoh}}}VV @VV{\pi_*}V \\
\mathrm{IndCoh}(\mathcal{Y}_2) @>{\Psi_{\mathcal{Y}_2}}>> \mathrm{QCoh}(\mathcal{Y}_2) \\
@V{\phi_*^{\mathrm{IndCoh}}}VV @VV{\phi_*}V \\
\mathrm{IndCoh}(\mathcal{Y}_3) @>{\Psi_{\mathcal{Y}_3}}>> \mathrm{QCoh}(\mathcal{Y}_3).
\end{CD}$$

3.7. **Direct image functor on** IndCoh**, further constructions.** The contents of this subsection will not be used elsewhere in the paper. We include it for completeness as the functor $\pi_{\text{non-ren},*}^{\mathrm{IndCoh}}$ introduced below has features analogous to those of the de Rham pushforward functor $\pi_{\mathrm{dR},*}$, considered in Sect. 7.4.

3.7.1. Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a morphism between arbitrary algebraic stacks. In this case also, we can introduce a functor $\mathrm{IndCoh}(\mathcal{Y}_1) \to \mathrm{IndCoh}(\mathcal{Y}_2)$, that we denote $\pi_{\text{non-ren},*}^{\mathrm{IndCoh}}$. This functor is *not necessarily continuous.*

By definition,

$$\pi_{\text{non-ren},*}^{\mathrm{IndCoh}}(\mathcal{F}) := \underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} (\pi \circ g)_*^{\mathrm{IndCoh}}(g^{\mathrm{IndCoh},*}(\mathcal{F})), \tag{3.25}$$

where $(\pi \circ g)_*^{\mathrm{IndCoh}}$ is well-defined because the morphism $\pi \circ g$ is schematic and quasi-compact.

3.7.2. Let us take for a moment $\mathcal{Y}_1 = \mathcal{Y}$ and $\mathcal{Y}_2 = \mathrm{pt}$. From (3.18) we obtain that

$$(p_{\mathcal{Y}})_{\text{non-ren},*}^{\mathrm{IndCoh}} \simeq \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, -). \tag{3.26}$$

3.7.3. Note that by construction we have a natural transformation

$$\Psi_{\mathcal{Y}_2} \circ \pi_{\text{non-ren},*}^{\mathrm{IndCoh}} \to \pi_* \circ \Psi_{\mathcal{Y}_1}. \tag{3.27}$$

We claim:

Lemma 3.7.4. *The natural transformation* (3.27) *is an isomorphism when applied to objects from* $\mathrm{IndCoh}(\mathcal{Y}_1)^+$.

Proof. Note that the functors $\Psi_{\mathcal{Y}_i}$ are t-exact, and both π_* and $\pi_{\text{non-ren},*}^{\mathrm{IndCoh}}$ are left t-exact. Hence, it is enough to show that the following diagram of functors commutes

$$\begin{CD}
\mathrm{IndCoh}(\mathcal{Y}_1)^{\geq n} @>{\Psi_{\mathcal{Y}_1}}>> \mathrm{QCoh}(\mathcal{Y}_1)^{\geq n} \\
@V{\pi_{\text{non-ren},*}^{\mathrm{IndCoh}}}VV @VV{\pi_*}V \\
\mathrm{IndCoh}(\mathcal{Y}_2)^{\geq n} @>{\Psi_{\mathcal{Y}_2}}>> \mathrm{QCoh}(\mathcal{Y}_2)^{\geq n}
\end{CD}$$

for every given n.

Note that $\Psi_{\mathcal{Y}_2}$, restricted to $\mathrm{IndCoh}(\mathcal{Y}_2)^{\geq n}$, is an equivalence, and hence commutes with limits. Hence, for $\mathcal{F}_1 \in \mathrm{IndCoh}(\mathcal{Y}_1)^{\geq n}$ we have:

$$\Psi_{\mathcal{Y}_2} \circ \pi^{\mathrm{IndCoh}}_{\text{non-ren},*} = \Psi_{\mathcal{Y}_2}\left(\underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} (\pi\circ g)^{\mathrm{IndCoh}}_*(g^{\mathrm{IndCoh},*}(\mathcal{F}))\right) \simeq$$
$$\simeq \underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} \Psi_{\mathcal{Y}_2}\left((\pi\circ g)^{\mathrm{IndCoh}}_*(g^{\mathrm{IndCoh},*}(\mathcal{F}))\right).$$

Since the morphisms $\pi\circ g$ are schematic and quasi-compact, and g is eventually coconnective, the latter expression can be rewritten as

$$\underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} ((\pi\circ g)_*(g^*(\Psi_{\mathcal{Y}_1}(\mathcal{F}_1)))),$$

which is isomorphic to $\pi_*(\Psi_{\mathcal{Y}_1}(\mathcal{F}_1))$ by Lemma 1.3.13, where we take the indexing category A to be $((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}}$.

□

3.7.5. Suppose for a moment that π is schematic and quasi-compact. It is easy to see that there exists a natural transformation

$$\pi^{\mathrm{IndCoh}}_* \to \pi^{\mathrm{IndCoh}}_{\text{non-ren},*}. \tag{3.28}$$

The next assertion can be proved by the same method as Proposition 7.5.4:

Proposition 3.7.6. *The natural transformation* (3.28) *is an isomorphism.*

3.7.7. It is easy to see that when π is eventually coconnective, the functor $\pi^{\mathrm{IndCoh}}_{\text{non-ren},*}$ is the right adjoint of $\pi^{\mathrm{IndCoh},*}$.

Remark 3.7.8. When π is not eventually coconnective, we do not know how to characterize the functor $\pi^{\mathrm{IndCoh}}_{\text{non-ren},*}$, except by the explicit formula (3.25).

3.7.9. Suppose that the morphism π is quasi-compact. Then it is easy to see that, that although the functor $\pi^{\mathrm{IndCoh}}_{\text{non-ren},*}$ is a priori non-continuous, it has has properties parallel to those of π_* expressed in Corollary 1.3.17(a,b): when restricted to $\mathrm{IndCoh}(\mathcal{Y}_1)^{\geq 0}$, it commutes with filtered colimits and is equipped with base change isomorphisms with respect to !-pullbacks for maps of algebraic stacks $\mathcal{Y}'_2 \to \mathcal{Y}_2$.

From the base change isomorphism for schematic quasi-compact maps we obtain that for a map $\phi_2 : \mathcal{Y}'_2 \to \mathcal{Y}_2$ and the corresponding Cartesian square

$$\begin{tikzcd} \mathcal{Y}'_1 \arrow[r, "\phi_1"] \arrow[d, "\pi'"'] & \mathcal{Y}_1 \arrow[d, "\pi"] \\ \mathcal{Y}'_2 \arrow[r, "\phi_2"] & \mathcal{Y}_2 \end{tikzcd} \tag{3.29}$$

there is a canonical natural transformation

$$\phi_2^! \circ \pi^{\mathrm{IndCoh}}_{\text{non-ren},*} \to \pi'^{\mathrm{IndCoh}}_{\text{non-ren},*} \circ \phi_1^!. \tag{3.30}$$

This natural natural transformation is not necessarily an isomorphism. But as we mentioned above, if π is quasi-compact, it is an isomorphism when applied to objects of $\mathrm{IndCoh}(\mathcal{Y}_1)^+$.

3.7.10. Suppose now that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are QCA. It is easy to see from the construction that there exists a canonical natural transformation

$$\pi_*^{\mathrm{IndCoh}} \to \pi_{\text{non-ren},*}^{\mathrm{IndCoh}}. \tag{3.31}$$

In Sect. 4.4.12 we will show:

Proposition 3.7.11. *The natural transformation* (3.31) *is an isomorphism.*

Remark 3.7.12. For $\mathcal{Y}_2 = \mathrm{pt}$, the assertion of Proposition 3.7.11 is easy: indeed, by (3.26) and (3.24), both functors identify canonically with $\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, -)$ of Sect. 3.6.2, where $\mathcal{Y} = \mathcal{Y}_1$.

From Proposition 3.7.11, we obtain:

Corollary 3.7.13. *If π is an eventually coconnective morphism between QCA stacks, the functors $(\pi^{\mathrm{IndCoh},*}, \pi_*^{\mathrm{IndCoh}})$ are adjoint.*

In addition, we have:

Corollary 3.7.14. *For a Cartesian square* (3.29) *there is a canonical isomorphism of functors*

$$\phi_2^! \circ \pi_*^{\mathrm{IndCoh}} \to \pi_*'^{\,\mathrm{IndCoh}} \circ \phi_1^!.$$

Proof. Both functors are continuous, so it is enough to construct the required natural transformation when restricted to the subcategory $\mathrm{Coh}(\mathcal{Y}_1)$. In this case, it follows from Proposition 3.7.11 and the isomorphism of (3.30) on $\mathrm{Coh}(\mathcal{Y}_1) \subset \mathrm{IndCoh}(\mathcal{Y}_1)^+$. $\square$

Remark 3.7.15. One can use Corollary 3.7.14 to define the functor π_*^{IndCoh} for QCA morphisms $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ between prestacks, in a way compatible with base change.

4. Dualizability and behavior with respect to products of stacks

In this section we will show that the category IndCoh on a QCA algebraic stack locally almost of finite type is dualizable, see Corollary 4.2.2. This will imply that the category $\mathrm{QCoh}(\mathcal{Y})$ on such a stack is also dualizable, under the additional assumption that $\mathcal{Y}$ be eventually coconnective, see Theorem 4.3.1.

These properties of $\mathrm{IndCoh}(\mathcal{Y})$ and $\mathrm{QCoh}(\mathcal{Y})$ will imply a "good" behavior of $\mathrm{IndCoh}(-)$ and $\mathrm{QCoh}(-)$ when we take a product of $\mathcal{Y}$ with another prestack.

In Sect. 4.4 we shall discuss applications to Serre duality on $\mathrm{IndCoh}(\mathcal{Y})$.

4.1. **The notion of dualizable DG category.**

4.1.1. *Definition of dualizability.* We refer to [Lu2], Sect. 6.3.2 for the definition of the tensor product functor

$$\otimes : \mathrm{DGCat}_{\mathrm{cont}} \times \mathrm{DGCat}_{\mathrm{cont}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

(see also [GL:DG], Sect. 1.4 for a brief review).

The above operation makes the $(\infty,1)$-category $\mathrm{DGCat}_{\mathrm{cont}}$ into a symmetric monoidal ∞-category [13], in which the unit object is the category Vect.

For an object of any symmetric monoidal category, one can talk about its property of being dualizable (see [Lu2], Sect. 4.2.5, or [GL:DG], Sect. 5.2 for a brief review). When the category is just monoidal, there are two different notions: left dualizable and right dualizable, see [GL:DG], Sect. 5.2.

[13] I.e., $\mathrm{DGCat}_{\mathrm{cont}}$ is a commutative algebra object in the symmetric monoidal $(1,\infty)$-category of ∞-categories with respect to the Cartesian product, see [Lu2], Sect. 2.3.1.

Remark 4.1.2. Note that dualizability of an object in not a higher-categorical notion, but only depends on the truncation of the monoidal ∞-category to an ordinary monoidal category.

Following Lurie, we say that $\mathbf{C} \in \mathrm{DGCat}_{\mathrm{cont}}$ is *dualizable* if it is dualizable in the above sense.

For $\mathbf{C} \in \mathrm{DGCat}_{\mathrm{cont}}$ dualizable, we denote by $\mathbf{C}^\vee$ the corresponding dual category. We denote by

$$\mathbf{C}^\vee \otimes \mathbf{C} \overset{\epsilon_{\mathbf{C}}}{\longrightarrow} \mathrm{Vect} \quad \text{and} \quad \mathrm{Vect} \overset{\mu_{\mathbf{C}}}{\longrightarrow} \mathbf{C} \otimes \mathbf{C}^\vee$$

the corresponding duality data. The functor $\epsilon_{\mathbf{C}}$ is called the *co-unit* of the pairing (or *evaluation*, or *canonical pairing*), and the functor $\mu_{\mathbf{C}}$ is called the *unit* (or, *co-evaluation*).

4.1.3. Here are some basic facts related to duality in $\mathrm{DGCat}_{\mathrm{cont}}$ (see also [GL:DG], Sect. 2):

(i) If $\mathbf{C}$ is dualizable, the category $\mathbf{C}^\vee$ can be recovered as $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}, \mathrm{Vect})$.

(ii) Any compactly generated DG category is dualizable.

(ii') For $\mathbf{C}$ compactly generated, $\mathbf{C}^\vee$ can be explicitly described as the ind-completion of the *non-cocomplete* DG category $(\mathbf{C}^c)^{\mathrm{op}}$. In particular, we have a canonical equivalence:

$$\mathbb{D}_{\mathbf{C}} : (\mathbf{C}^\vee)^c \simeq (\mathbf{C}^c)^{\mathrm{op}}.$$

In particular, for $\mathbf{C} = \mathrm{Ind}(\mathbf{C}^0)$ (see Sect. 0.6.7), we have $\mathbf{C}^\vee \simeq \mathrm{Ind}((\mathbf{C}^0)^{\mathrm{op}})$, and

$$\mathbf{C}^\vee \simeq \mathrm{Funct}(\mathbf{C}^0, \mathrm{Vect}) \text{ and } \mathbf{C} \simeq \mathrm{Funct}((\mathbf{C}^0)^{\mathrm{op}}, \mathrm{Vect}),$$

which also gives an explicit construction of $\mathrm{Ind}(\mathbf{C}^0)$.

(iii) The functor of tensoring by a dualizable category commutes with all limits[14] taken in $\mathrm{DGCat}_{\mathrm{cont}}$. Indeed, if $\mathbf{C}$ is dualizable then $\mathbf{C} \otimes - \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}^\vee, -)$.

4.1.4. Let $\mathbf{O}$ be an arbitrary symmetric monoidal category, and $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{O}$ two dualizable objects. Then to any morphism $f : \mathbf{c}_1 \to \mathbf{c}_2$ one canonically attaches the dual morphism

$$f^\vee : \mathbf{c}_2^\vee \to \mathbf{c}_1^\vee,$$

where $\mathbf{c}_i^\vee$ denotes the dual of $\mathbf{c}_i$.

This construction has the following interpretation: a datum morphism f as above is equivalent to that of a point in $\mathrm{Maps}_{\mathbf{O}}(1, \mathbf{c}_1^\vee \otimes \mathbf{c}_2)$. Then the datum $f^\vee$ corresponds to *the same* point in

$$\mathrm{Maps}_{\mathbf{O}}(1, (\mathbf{c}_2^\vee)^\vee \otimes \mathbf{c}_1^\vee) \simeq \mathrm{Maps}_{\mathbf{O}}(1, \mathbf{c}_1^\vee \otimes \mathbf{c}_2).$$

Applying this to $\mathbf{O} = \mathrm{DGCat}_{\mathrm{cont}}$ and two dualizable categories $\mathbf{C}_1$ and $\mathbf{C}_2$, we obtain that to every continuous functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ there corresponds a dual functor

$$F^\vee : \mathbf{C}_2^\vee \to \mathbf{C}_1^\vee.$$

In terms of Sect. 4.1.3(i), the functor $F^\vee$ can be described as follows: it sends an object $\Phi \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_2, \mathrm{Vect})$ to $\Phi \circ F \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_1, \mathrm{Vect})$.

4.2. **Dualizability of** IndCoh**.**

[14] Tensoring by $\mathbf{C}$ commutes with all colimits in $\mathrm{DGCat}_{\mathrm{cont}}$ for any $\mathbf{C}$.

4.2.1. From Sect. 4.1.3(ii) and Theorem 3.3.5 we obtain:

Corollary 4.2.2. *If $\mathcal{Y}$ is a QCA algebraic stack, then the DG category* $\operatorname{IndCoh}(\mathcal{Y})$ *is dualizable.*

As was explained to us by J. Lurie, Corollary 4.2.2 implies the following result (in any sheaf-theoretic context):

Corollary 4.2.3. *Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be two prestacks, with $\mathcal{Y}_1$ being a QCA algebraic stack. Then the natural functor*

$$\operatorname{IndCoh}(\mathcal{Y}_1) \otimes \operatorname{IndCoh}(\mathcal{Y}_2) \to \operatorname{IndCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

is an equivalence.

Proof. The argument repeats verbatim that of [GL:QCoh, Proposition 1.4.4]. For completeness, let us reproduce it here:

We will show that the equivalence stated in the corollary takes place for any two prestacks $\mathcal{Y}_1$, $\mathcal{Y}_2$, whenever $\operatorname{IndCoh}(\mathcal{Y}_1)$ is dualizable.

We have:

$$\operatorname{IndCoh}(\mathcal{Y}_1) \otimes \operatorname{IndCoh}(\mathcal{Y}_2) = \operatorname{IndCoh}(\mathcal{Y}_1) \otimes \left(\underset{S_2 \in ((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \operatorname{IndCoh}(S_2) \right).$$

By Sect. 4.1.3(iii), the latter expression maps isomorphically to

$$\underset{S_2 \in ((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \left(\operatorname{IndCoh}(\mathcal{Y}_1) \otimes \operatorname{IndCoh}(S_2) \right).$$

We rewrite $\operatorname{IndCoh}(\mathcal{Y}_1)$ by definition as

$$\underset{S_1 \in ((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \operatorname{IndCoh}(S_1),$$

so

$$\underset{S_2 \in ((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \left(\operatorname{IndCoh}(\mathcal{Y}_1) \otimes \operatorname{IndCoh}(S_2) \right) \simeq$$

$$\simeq \underset{S_2 \in ((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \left(\left(\underset{S_1 \in ((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \operatorname{IndCoh}(S_1) \right) \otimes \operatorname{IndCoh}(S_2) \right).$$

Since $\operatorname{IndCoh}(S_2)$ is dualizable, by Sect. 4.1.3(iii), the latter expression can be rewritten as

$$\underset{S_2 \in ((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \left(\underset{S_1 \in ((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \left(\operatorname{IndCoh}(S_1) \otimes \operatorname{IndCoh}(S_2) \right) \right). \tag{4.1}$$

Now, as was mentioned in Sect. 3.2.3, for quasi-compact schemes S_1 and S_2, the natural functor

$$\operatorname{IndCoh}(S_1) \otimes \operatorname{IndCoh}(S_2) \to \operatorname{IndCoh}(S_1 \times S_2)$$

is an equivalence.

Hence, we obtain that the expression in (4.1) maps isomorphically to

$$\underset{S_2\in((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2})^{\mathrm{op}}}{\underleftarrow{lim}} \left(\underset{S_1\in((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1})^{\mathrm{op}}}{\underleftarrow{lim}} (\mathrm{IndCoh}(S_1\times S_2))\right),$$

which itself is isomorphic to

$$\underset{(S_1,S_2)\in((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1})^{\mathrm{op}}\times((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2})^{\mathrm{op}}}{\underleftarrow{lim}} (\mathrm{IndCoh}(S_1\times S_2)).$$

To summarize, we obtain an equivalence

$$\begin{aligned}(4.2)\quad &\mathrm{IndCoh}(\mathcal{Y}_1)\otimes\mathrm{IndCoh}(\mathcal{Y}_2)\to \\ &\to \underset{(S_1,S_2)\in((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1})^{\mathrm{op}}\times((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2})^{\mathrm{op}}}{\underleftarrow{lim}} (\mathrm{IndCoh}(S_1\times S_2)).\end{aligned}$$

Finally, it is easy to see that the natural functor

$$(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1}\times(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2}\to(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1\times\mathcal{Y}_2}$$

is cofinal. Hence, the functor

$$\begin{aligned}\mathrm{IndCoh}(\mathcal{Y}_1\times\mathcal{Y}_2) = &\underset{S\in((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1\times\mathcal{Y}_2})^{\mathrm{op}}}{\underleftarrow{lim}} \mathrm{IndCoh}(S)\to \\ &\to \underset{(S_1,S_2)\in((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1})^{\mathrm{op}}\times((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2})^{\mathrm{op}}}{\underleftarrow{lim}} (\mathrm{IndCoh}(S_1\times S_2))\end{aligned}$$

is an equivalence, and the composition

$$\begin{aligned}\mathrm{IndCoh}(\mathcal{Y}_1)\otimes\mathrm{IndCoh}(\mathcal{Y}_2)\to\mathrm{IndCoh}(\mathcal{Y}_1\times\mathcal{Y}_2) = &\underset{S\in((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1\times\mathcal{Y}_2})^{\mathrm{op}}}{\underleftarrow{lim}} \mathrm{IndCoh}(S)\to \\ &\to \underset{(S_1,S_2)\in((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_1})^{\mathrm{op}}\times((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2})^{\mathrm{op}}}{\underleftarrow{lim}} (\mathrm{IndCoh}(S_1\times S_2))\end{aligned}$$

is the map (4.2).

This proves that the map $\mathrm{IndCoh}(\mathcal{Y}_1)\otimes\mathrm{IndCoh}(\mathcal{Y}_2)\to\mathrm{IndCoh}(\mathcal{Y}_1\times\mathcal{Y}_2)$ is an equivalence. □

4.3. **Applications to** $\mathrm{QCoh}(\mathcal{Y})$**.**
We will now use Corollary 4.2.2 to prove the following:

Theorem 4.3.1. *Let $\mathcal{Y}$ be a QCA algebraic stack, which is eventually coconnective (see Definition 1.4.8), and locally almost of finite type (as are all algebraic stacks in this section). Then the category* $\mathrm{QCoh}(\mathcal{Y})$ *is dualizable.*

Remark 4.3.2. We do not know whether, under the assumptions of the theorem, the category $\mathrm{QCoh}(\mathcal{Y})$ is compactly generated.

Proof. Recall (see [IndCoh, Sect. 11.7.3]) that for any eventually coconnective algebraic stack $\mathcal{Y}$, the functor $\Psi_{\mathcal{Y}}:\mathrm{IndCoh}(\mathcal{Y})\to\mathrm{QCoh}(\mathcal{Y})$ admits a left adjoint, which is fully faithful (and automatically continuous by virtue of being a left adjoint).

In particular, we obtain that in this case, $\mathrm{QCoh}(\mathcal{Y})$ is a *retract* of $\mathrm{IndCoh}(\mathcal{Y})$ in the category $\mathrm{DGCat}_{\mathrm{cont}}$.

The assertion of the theorem follows from the following observation: let $\mathbf{O}$ be a monoidal category, which admits inner Hom's, i.e., for $M_1, M_2 \in \mathbf{O}$, there exists an object

$$\underline{\mathrm{Hom}}_{\mathbf{O}}(M_1, M_2) \in \mathbf{O},$$

such that we have

$$\mathrm{Maps}_{\mathbf{O}}(N, \underline{\mathrm{Hom}}_{\mathbf{O}}(M_1, M_2)) \simeq \mathrm{Maps}_{\mathbf{O}}(N \otimes M_1, M_2),$$

functorially in N.

Lemma 4.3.3. *Under the above circumstances, a retract of a (left) dualizable object is (left) dualizable.*

Proof. It is easy to see that an object M is (left) dualizable if and only if for any N, the natural map

$$N \otimes \underline{\mathrm{Hom}}_{\mathbf{O}}(M, 1) \to \mathrm{Maps}(N, M)$$

is an isomorphism. However, the latter condition survives taking retracts. □

We apply this lemma to $\mathbf{O} = \mathrm{DGCat}_{\mathrm{cont}}$. This category has inner Hom's, which are explicitly given by

$$\underline{\mathrm{Hom}}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathbf{C}_1, \mathbf{C}_2) = \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_1, \mathbf{C}_2),$$

where the right-hand side has a natural structure of DG category.

□

Corollary 4.3.4. *Let $\mathcal{Y}$ satisfy the assumptions of Theorem 4.3.1. Then for any prestack $\mathcal{Y}'$, the natural functor*

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}')$$

is an equivalence.

Proof. This follows from Theorem 4.3.1 by [GL:QCoh, Proposition 1.4.4], which repeats verbatim the proof of Corollary 4.2.3. □

Remark 4.3.5. The assertion of Corollary 4.3.4, together with the proof, is valid for *all* prestacks $\mathcal{Y}'$, i.e., not necessarily those locally almost of finite type.

4.3.6. Let us recall the notion of rigid monoidal DG category from [GL:DG], Sect. 6.1. This notion can be formulated as follows: a monoidal category $\mathbf{O}$ is rigid if:

- The object $1 \in \mathbf{O}$ is compact.
- The functor

$$\mathbf{O} \to \mathbf{O} \otimes \mathbf{O}, \tag{4.3}$$

  right adjoint to $\mathbf{O} \otimes \mathbf{O} \xrightarrow{\otimes} \mathbf{O}$, is continuous, and is compatible with left and right actions of $\mathbf{O}$.

If this happens, the functors

$$\mathbf{O} \otimes \mathbf{O} \xrightarrow{\otimes} \mathbf{O} \xrightarrow{\mathcal{M}aps_{\mathbf{O}}(1,-)} \mathrm{Vect}$$

and

$$\mathrm{Vect} \to \mathbf{O} \to \mathbf{O} \otimes \mathbf{O},$$

(where the functor $\mathrm{Vect} \to \mathbf{O}$ is given by $1 \in \mathbf{O}$, and the functor $\mathbf{O} \to \mathbf{O} \otimes \mathbf{O}$ is (4.3)) define a duality datum between $\mathbf{O}$ and itself.

4.3.7. We have:

Corollary 4.3.8. *Let $\mathcal{Y}$ be as in Theorem 4.3.1. Then the monoidal category* $\mathrm{QCoh}(\mathcal{Y})$ *is rigid.*

Proof. This is [GL:QCoh, Proposition 2.3.2]: the assertion is true for any prestack (not necessarily of finite type) with the following three properties: (1) the category $\mathrm{QCoh}(\mathcal{Y})$ is dualizable, (2) the object $\mathcal{O}_\mathcal{Y} \in \mathrm{QCoh}(\mathcal{Y})$ is compact, and (3) the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is schematic, quasi-separated and quasi-compact. □

In particular, we obtain a canonical identification

$$\mathbf{D}^{\mathrm{naive}}_\mathcal{Y} : \mathrm{QCoh}(\mathcal{Y})^\vee \simeq \mathrm{QCoh}(\mathcal{Y}),$$

where the duality datum is described as follows:

The functor $\epsilon_{\mathrm{QCoh}(\mathcal{Y})}$ is given by

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \overset{\boxtimes}{\longrightarrow} \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}) \overset{\Delta^*}{\longrightarrow} \mathrm{QCoh}(\mathcal{Y}) \overset{\Gamma(\mathcal{Y},-)}{\longrightarrow} \mathrm{Vect},$$

and the functor $\mu_{\mathrm{QCoh}(\mathcal{Y})}$ is given by

$$\mathrm{Vect} \overset{\mathcal{O}_\mathcal{Y}}{\longrightarrow} \mathrm{QCoh}(\mathcal{Y}) \overset{\Delta_*}{\longrightarrow} \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}) \simeq \mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}).$$

4.4. **Serre duality on** $\mathrm{IndCoh}(\mathcal{Y})$**.**

4.4.1. Recall (see [IndCoh, Sect. 9.2.1]) that for a quasi-compact DG scheme Z, there exists a canonical involutive equivalence:

$$\mathbf{D}^{\mathrm{Serre}}_Z : \mathrm{IndCoh}(Z)^\vee \simeq \mathrm{IndCoh}(Z).$$

In terms of Sect. 4.1.3(ii'), the above equivalence corresponds to the identification

$$(\mathrm{IndCoh}(Z)^c)^{\mathrm{op}} = \mathrm{Coh}(Z)^{\mathrm{op}} \overset{\mathbb{D}^{\mathrm{Serre}}_Z}{\longrightarrow} \mathrm{Coh}(Z) = \mathrm{IndCoh}(Z)^c,$$

where the middle arrow is the *Serre duality* functor. Explicitly, for $\mathcal{F} \in \mathrm{Coh}(Z)$,

$$\mathbb{D}^{\mathrm{Serre}}_Z(\mathcal{F}) = \underline{\mathrm{Hom}}_{\mathrm{QCoh}(Z)}(\mathcal{F}, \omega_Z),$$

which is a priori an object of $\mathrm{QCoh}(Z)$, but in fact can be easily shown to belong to $\mathrm{Coh}(Z)$.

Remark 4.4.2. In the above formula, $\underline{\mathrm{Hom}}_{\mathrm{QCoh}(Z)}(-,-)$ denotes the inner Hom of [GL:DG], Sect. 5.1, defined whenever a monoidal category (in our case $\mathrm{QCoh}(Z)$) is acting on a module category (in our case $\mathrm{IndCoh}(Z)$).

Our current goal is to show that the same goes through, when instead of a quasi-compact DG scheme Z we have a QCA algebraic stack $\mathcal{Y}$.

4.4.3. First, let $\mathcal{Y}$ be any algebraic stack. Recall the (non-cocomplete) category $\mathrm{Coh}(\mathcal{Y})$, see Sect. 3.3. We obtain that there exists a canonical equivalence:

$$\mathbb{D}^{\mathrm{Serre}}_\mathcal{Y} : \mathrm{Coh}(\mathcal{Y})^{\mathrm{op}} \overset{\sim}{\longrightarrow} \mathrm{Coh}(\mathcal{Y}), \tag{4.4}$$

characterized by the property that for every affine (or quasi-compact) quasi-compact DG scheme S equipped with a *smooth* map $g : S \to \mathcal{Y}$, we have an identification

$$g^{\mathrm{IndCoh},*} \circ \mathbb{D}^{\mathrm{Serre}}_\mathcal{Y} \simeq \mathbb{D}^{\mathrm{Serre}}_S \circ (g^!)^{\mathrm{op}},$$

as functors $\mathrm{Coh}(\mathcal{Y})^{\mathrm{op}} \to \mathrm{Coh}(Z)$. Moreover, $\mathbb{D}^{\mathrm{Serre}}_\mathcal{Y}$ is naturally involutive.

Proposition 4.4.4. *For $\mathcal{F}_1 \in \mathrm{Coh}(\mathcal{Y})^{\mathrm{op}}$ and $\mathcal{F}_2 \in \mathrm{IndCoh}(\mathcal{Y})$ we have a canonical isomorphism*

$$\mathcal{M}aps(\mathbb{D}^{\mathrm{Serre}}_{\mathcal{Y}}(\mathcal{F}_1), \mathcal{F}_2) \simeq \Gamma^{\mathrm{IndCoh}}\left(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2\right). \tag{4.5}$$

Remark 4.4.5. The assertion of the proposition when $\mathcal{Y}$ is a quasi-compact DG scheme Z follows from the definition of the evaluation map

$$\mathrm{IndCoh}(Z) \otimes \mathrm{IndCoh}(Z) \to \mathrm{Vect},$$

see [IndCoh, Sect. 9.2.2].

Proof. The left-hand side in (4.5) identifies with

$$\underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \mathcal{M}aps_{\mathrm{Coh}(Z)}\left(g^!(\mathbb{D}^{\mathrm{Serre}}_{\mathcal{Y}}(\mathcal{F}_1)), g^!(\mathcal{F}_2)\right). \tag{4.6}$$

One can rewrite $\mathcal{M}aps_{\mathrm{Coh}(S)}\left(g^!(\mathbb{D}^{\mathrm{Serre}}_{\mathcal{Y}}(\mathcal{F}_1)), g^!(\mathcal{F}_2)\right)$ as

$$\mathcal{M}aps_{\mathrm{Coh}(S)}\left(\mathbb{D}^{\mathrm{Serre}}_{Z}(g^{\mathrm{IndCoh},*}(\mathcal{F}_1)), g^!(\mathcal{F}_2)\right) \simeq \Gamma^{\mathrm{IndCoh}}\left(S, g^{\mathrm{IndCoh},*}(\mathcal{F}_1) \overset{!}{\otimes} g^!(\mathcal{F}_2)\right),$$

where the last isomorphism takes place because of Remark 4.4.5.

Note that for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y})$, by (3.18) have:

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, \mathcal{F}) \simeq \underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \Gamma^{\mathrm{IndCoh}}\left(S, g^{\mathrm{IndCoh},*}(\mathcal{F})\right).$$

Therefore, the right-hand side in (4.5) is canonically isomorphic to

$$\underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \Gamma^{\mathrm{IndCoh}}\left(S, g^{\mathrm{IndCoh},*}(\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2)\right).$$

Therefore, in order to construct the isomorphism in (4.5), it remains to construct a compatible family of isomorphisms of functors

$$\Delta^!_S \circ (g^{\mathrm{IndCoh},*} \boxtimes g^!) \simeq g^{\mathrm{IndCoh},*} \circ \Delta^!_{\mathcal{Y}}. \tag{4.7}$$

The latter isomorphism of functors is valid for any k-representable, eventually coconnective morphism between prestacks $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$: it follows by applying (3.11) to the Cartesian diagram

$$\begin{array}{ccc} \mathcal{Y}_1 & \longrightarrow & \mathcal{Y}_2 \times \mathcal{Y}_1 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{Y}_2} \times \pi} \\ \mathcal{Y}_2 & \xrightarrow{\Delta_{\mathcal{Y}_2}} & \mathcal{Y}_2 \times \mathcal{Y}_2. \end{array}$$

$\square$

4.4.6. Assume now that $\mathcal{Y}$ is a QCA algebraic stack. Then by Theorem 3.3.5,

$$\mathrm{IndCoh}(\mathcal{Y}) \simeq \mathrm{Ind}(\mathrm{Coh}(\mathcal{Y})).$$

So, by Sect. 4.1.3(ii'), from (4.4) we deduce:

Corollary 4.4.7. *For a QCA algebraic stack $\mathcal{Y}$ there is a natural involutive identification:*

$$\mathbf{D}^{\mathrm{Serre}}_{\mathcal{Y}} : \mathrm{IndCoh}(\mathcal{Y})^{\vee} \simeq \mathrm{IndCoh}(\mathcal{Y}). \tag{4.8}$$

4.4.8. Will shall now describe explicitly the duality data $\epsilon_{\mathrm{IndCoh}(\mathcal{Y})}$ and $\mu_{\mathrm{IndCoh}(\mathcal{Y})}$ that corresponds to the equivalence (4.8). We claim:

Proposition 4.4.9. *Let $\mathcal{Y}$ be a QCA algebraic stack. Then the duality* (4.8) *has as evaluation $\epsilon_{\mathrm{IndCoh}(\mathcal{Y})}$ the functor*

$$\mathrm{IndCoh}(\mathcal{Y})\otimes\mathrm{IndCoh}(\mathcal{Y})\to\mathrm{IndCoh}(\mathcal{Y}\times\mathcal{Y})\overset{\Delta^!_{\mathcal{Y}}}{\longrightarrow}\mathrm{IndCoh}(\mathcal{Y})\overset{\Gamma^{\mathrm{IndCoh}}(\mathcal{Y},-)}{\longrightarrow}\mathrm{Vect},\tag{4.9}$$

and as a co-evaluation $\mu_{\mathrm{IndCoh}(\mathcal{Y})}$ the functor

$$\mathrm{Vect}\overset{\omega_{\mathcal{Y}}\otimes-}{\longrightarrow}\mathrm{IndCoh}(\mathcal{Y})\overset{(\Delta_{\mathcal{Y}})^{\mathrm{IndCoh}}_*}{\longrightarrow}\mathrm{IndCoh}(\mathcal{Y}\times\mathcal{Y})\simeq\mathrm{IndCoh}(\mathcal{Y})\otimes\mathrm{IndCoh}(\mathcal{Y}).\tag{4.10}$$

Proof. Let $\mathcal{F}_1,\mathcal{F}_2$ be two objects of $\mathrm{Coh}(\mathcal{Y})$. In order to identity $\epsilon_{\mathrm{IndCoh}(\mathcal{Y})}$ with the functor (4.9), we need to establish a functorial isomorphism

$$\mathcal{M}aps_{\mathrm{Coh}(\mathcal{Y})}(\mathbb{D}^{\mathrm{Serre}}_{\mathcal{Y}}(\mathcal{F}_1),\mathcal{F}_2)\simeq\Gamma^{\mathrm{IndCoh}}(\mathcal{Y},\mathcal{F}_1\overset{!}{\otimes}\mathcal{F}_2).$$

However, this is the content of Proposition 4.4.4.

In order to prove that $\mu_{\mathrm{IndCoh}(\mathcal{Y})}$ is given by (4.10), it is sufficient to show that the composition

$$\mathrm{IndCoh}(\mathcal{Y})\overset{\mathrm{Id}_{\mathrm{IndCoh}(\mathcal{Y})}\otimes(4.10)}{\longrightarrow}\mathrm{IndCoh}(\mathcal{Y})\otimes\mathrm{IndCoh}(\mathcal{Y})\otimes\mathrm{IndCoh}(\mathcal{Y})\overset{(4.9)\otimes\mathrm{Id}_{\mathrm{IndCoh}(\mathcal{Y})}}{\longrightarrow}\mathrm{IndCoh}(\mathcal{Y})$$

is isomorphic to the identity functor.

Consider the diagram

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\Delta_{\mathcal{Y}}"] \arrow[d, "\Delta_{\mathcal{Y}}"'] & \mathcal{Y}\times\mathcal{Y} \arrow[r, "\mathrm{id}\times p_{\mathcal{Y}}"] \arrow[d, "\mathrm{id}\times\Delta_{\mathcal{Y}}"] & \mathcal{Y} \\
\mathcal{Y}\times\mathcal{Y} \arrow[r, "\Delta_{\mathcal{Y}}\times\mathrm{id}"] \arrow[d, "p_{\mathcal{Y}}\times\mathrm{id}"'] & \mathcal{Y}\times\mathcal{Y}\times\mathcal{Y} & \\
\mathcal{Y}. & &
\end{tikzcd}$$

We need to show that the functor

$$(\mathrm{Id}_{\mathrm{IndCoh}(\mathcal{Y})}\otimes(p_{\mathcal{Y}})^{\mathrm{IndCoh}}_*)\circ(\mathrm{id}\times\Delta_{\mathcal{Y}})^!\circ(\Delta_{\mathcal{Y}}\times\mathrm{id})^{\mathrm{IndCoh}}_*\circ(p^!_{\mathcal{Y}}\otimes\mathrm{Id}_{\mathrm{IndCoh}(\mathcal{Y})})\tag{4.11}$$

is isomorphic to the identity functor.

However, in the above diagram the inner square is Cartesian and the arrows in it are schematic and quasi-compact. Therefore, by the base change isomorphism, we have

$$(\Delta_{\mathcal{Y}})^{\mathrm{IndCoh}}_*\circ(\Delta_{\mathcal{Y}})^!\simeq(\mathrm{id}\times\Delta_{\mathcal{Y}})^!\circ(\Delta_{\mathcal{Y}}\times\mathrm{id})^{\mathrm{IndCoh}}_*:\mathrm{IndCoh}(\mathcal{Y}\times\mathcal{Y})\to\mathrm{IndCoh}(\mathcal{Y}\times\mathcal{Y}).$$

Therefore, the functor in (4.11) is isomorphic to

$$\begin{aligned}(\mathrm{Id}_{\mathrm{IndCoh}(\mathcal{Y})}\otimes(p_{\mathcal{Y}})^{\mathrm{IndCoh}}_*)\circ(\Delta_{\mathcal{Y}})^{\mathrm{IndCoh}}_*\circ(\Delta_{\mathcal{Y}})^!\circ(p^!_{\mathcal{Y}}\otimes\mathrm{Id}_{\mathrm{IndCoh}(\mathcal{Y})})\simeq\\ \simeq(\mathrm{id}\times p_{\mathcal{Y}})_*\circ(\Delta_{\mathcal{Y}})^{\mathrm{IndCoh}}_*\circ(\Delta_{\mathcal{Y}})^!\circ(p_{\mathcal{Y}}\times\mathrm{id})^!\simeq\\ \simeq((\mathrm{id}\times p_{\mathcal{Y}})\circ\Delta_{\mathcal{Y}})^{\mathrm{IndCoh}}_*\circ(\Delta_{\mathcal{Y}}\circ(p_{\mathcal{Y}}\times\mathrm{id}))^!\simeq(\mathrm{id})^{\mathrm{IndCoh}}_*\circ\mathrm{id}^!\simeq\mathrm{Id}\,.\end{aligned}$$

□

4.4.10. Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of QCA algebraic stacks. We have the functors

$$\pi_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{Y}_1) \to \mathrm{IndCoh}(\mathcal{Y}_2) \text{ and } \pi^! : \mathrm{IndCoh}(\mathcal{Y}_2) \to \mathrm{IndCoh}(\mathcal{Y}_1).$$

We claim that these functors are related as follows. Recall the notion of dual functor, see Sect. 4.1.4.

Proposition 4.4.11. *Under the identifications* $\mathbf{D}^{\mathrm{Serre}}_{\mathcal{Y}_i} : \mathrm{IndCoh}(\mathcal{Y}_i)^\vee \simeq \mathrm{IndCoh}(\mathcal{Y}_i)$, *we have:*

$$(\pi_*^{\mathrm{IndCoh}})^\vee \simeq \pi^!.$$

Proof. We need to show that the object in

$$\mathrm{IndCoh}(\mathcal{Y}_1)^\vee \otimes \mathrm{IndCoh}(\mathcal{Y}_2) \simeq \mathrm{IndCoh}(\mathcal{Y}_1) \otimes \mathrm{IndCoh}(\mathcal{Y}_2) \simeq \mathrm{IndCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

that corresponds to π_*^{IndCoh} is isomorphic to the object that corresponds to $\pi^!$. The former is given by

$$(\mathrm{id}_{\mathcal{Y}_1} \times \pi)_*^{\mathrm{IndCoh}} \circ (\Delta_{\mathcal{Y}_1})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}_1}),$$

and the latter by

$$(\pi \times \mathrm{id}_{\mathcal{Y}_2})^! \circ (\Delta_{\mathcal{Y}_2})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}_2}).$$

The needed isomorphism follows by base change (see Sect. 3.2.9) from the Cartesian diagram

$$\begin{CD}
\mathcal{Y}_1 @>{\mathrm{Graph}(\pi)}>> \mathcal{Y}_1 \times \mathcal{Y}_2 \\
@V{\pi}VV @VV{\pi \times \mathrm{id}_{\mathcal{Y}_2}}V \\
\mathcal{Y}_2 @>{\Delta_{\mathcal{Y}_2}}>> \mathcal{Y}_2 \times \mathcal{Y}_2,
\end{CD}$$

in which the horizontal arrows are schematic and quasi-compact. □

4.4.12. *Proof of Proposition 3.7.11.* As was mentioned in Remark 3.7.12, we note that the assertion of the proposition when $\mathcal{Y}_2 = \mathrm{pt}$ is the isomorphism (3.18).

In the general case, by Proposition 4.4.11, it suffices to show that for $\mathcal{F}_2 \in \mathrm{Coh}(\mathcal{Y}_2)$ and $\mathcal{F}_1 \in \mathrm{IndCoh}(\mathcal{Y}_1)$ the natural map

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}_1, \mathcal{F}_1 \overset{!}{\otimes} \pi^!(\mathcal{F}_2)) \to \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}_2, \pi^{\mathrm{IndCoh}}_{\text{non-ren},*}(\mathcal{F}_1) \overset{!}{\otimes} \mathcal{F}_2) \tag{4.12}$$

is an isomorphism. We rewrite the right-hand side as

$$\mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{F}_2)}(\mathbb{D}^{\mathrm{Serre}}_{\mathcal{Y}_2}(\mathcal{F}_2), \pi^{\mathrm{IndCoh}}_{\text{non-ren},*}(\mathcal{F}_1)),$$

and further as

$$\underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} \mathcal{M}aps_{\mathrm{IndCoh}(\mathcal{F}_2)}\left(\mathbb{D}^{\mathrm{Serre}}_{\mathcal{Y}_2}(\mathcal{F}_2), (\pi \circ g)_*^{\mathrm{IndCoh}}(g^{\mathrm{IndCoh},*}(\mathcal{F}_1))\right).$$

The latter expression can be rewritten as

$$\underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}_2, (\pi \circ g)_*^{\mathrm{IndCoh}}(g^{\mathrm{IndCoh},*}(\mathcal{F}_1)) \overset{!}{\otimes} \mathcal{F}_2) \simeq$$

$$\simeq \underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} \Gamma^{\mathrm{IndCoh}}(S, g^{\mathrm{IndCoh},*}(\mathcal{F}_1) \overset{!}{\otimes} (\pi \circ g)^!(\mathcal{F}_2)).$$

Using the fact that

$$g^{\mathrm{IndCoh},*}(\mathcal{F}_1) \overset{!}{\otimes} (\pi \circ g)^!(\mathcal{F}_2) \simeq g^{\mathrm{IndCoh},*}(\mathcal{F}_1 \overset{!}{\otimes} \pi^!(\mathcal{F}_2))$$

(see (4.7)), we finally obtain that the right-hand side in (4.12) is isomorphic to

$$\underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \Gamma^{\mathrm{IndCoh}}\left(S, g^{\mathrm{IndCoh},*}(\mathcal{F}_1 \overset{!}{\otimes} \pi^!(\mathcal{F}_2))\right) \simeq \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}_1, \mathcal{F}_1 \overset{!}{\otimes} \pi^!(\mathcal{F}_2)),$$

as required.

□

5. Recollections: D-modules on DG schemes

This section is devoted to a review of the theory of D-modules on (DG) schemes. As was mentioned in the introduction, this material is well-known at the level of triangulated categories. However, no comprehensive account seems to exist at the DG level.[15]

We remind that according to the conventions of Sect. 3.1, *all DG schemes, algebraic stacks and prestacks are assumed locally almost of finite type, unless specified otherwise.*

5.1. **The basics.**

5.1.1. To any quasi-compact DG scheme[16] Z one assigns the category D-mod(Z) of right D-modules on Z.

5.1.2. By definition,

$$\text{D-mod}(Z) := \mathrm{IndCoh}(Z_{\mathrm{dR}}),$$

see [GR1, Sect. 2.3.2]. (Note that in *loc.cit*, the category D-mod(Z) is denoted $\mathrm{Crys}^r(Z)$.)

In the above formula Z_{dR} is the de Rham prestack of Z, i.e.,

$$\mathrm{Maps}(S, Z_{\mathrm{dR}}) := \mathrm{Maps}((^{cl}S)_{red}, Z),$$

where $(^{cl}S)_{red}$ denotes the classical reduced scheme underlying S, see [GR1, Sect. 1.1.1].

5.1.3. For any map $f : Z_1 \to Z_2$ of quasi-compact DG schemes, there exists a canonically defined continuous functor

$$f^! : \text{D-mod}(Z_2) \to \text{D-mod}(Z_1).$$

If f is proper[17], the functor $f^!$ admits a *left* adjoint, denoted $f_{\mathrm{dR},*}$. If f is an open embedding, the functor $f^!$ admits a continuous *right* adjoint, also denoted $f_{\mathrm{dR},*}$.

[15] That said, the "local" aspects of the theory of D-modules (i.e., when we only need to pull back, but not push forward) is a formal consequence of IndCoh by the procedure of passage to the de Rham prestack. Details on that can be found in [GR1].

[16] According to Sect. 5.1.10 below, D-mod(Z) depends only on the underlying classical scheme ^{cl}Z. The only reason for working in the format of DG schemes is that we will discuss the relation between D-mod(Z) and the category IndCoh(Z), which depends on the DG structure.

[17] A morphism of DG schemes is said to be proper if the underlying morphism of classical schemes is.

5.1.4. *Descent.* The assignment $Z \rightsquigarrow \text{D-mod}(Z)$ satisfies fppf descent.

In particular, it satisfies Zariski descent, so the category $\text{D-mod}(Z)$ is glued from the categories $\text{D-mod}(U_i)$, where $\{U_i\}$ is a Zariski-open affine cover of Z.

Therefore, for many purposes it is sufficient to consider the case of affine DG schemes.

In addition, gluing can be used to define $\text{D-mod}(Z)$ on a not necessarily quasi-compact DG scheme, as well as the functor $f^! : \text{D-mod}(Z_2) \to \text{D-mod}(Z_1)$ for a map $f : Z_1 \to Z_2$ of not necessarily quasi-compact DG schemes.

This will be a particular case of the definition of $\text{D-mod}(\mathcal{Y})$ on a prestack $\mathcal{Y}$, see Sect. 6.1.1.

5.1.5. *Relation between* $\text{D-mod}(Z)$ *and* $\text{IndCoh}(Z)$*.* For a DG scheme Z we have a pair of mutually adjoint (continuous) functors

$$\mathbf{ind}_{\text{D-mod}(Z)} : \text{IndCoh}(Z) \rightleftarrows \text{D-mod}(Z) : \mathbf{oblv}_{\text{D-mod}(Z)},$$

with $\mathbf{oblv}_{\text{D-mod}(Z)}$ being conservative. The functor $\mathbf{oblv}_{\text{D-mod}(Z)}$ corresponds to pullback along the tautological morphism $Z \to Z_{\text{dR}}$.

For a morphism of DG schemes $f : Z_1 \to Z_2$, we have a commutative diagram

$$\begin{tikzcd}
\text{IndCoh}(Z_1) & \text{D-mod}(Z_1) \arrow[l, "\mathbf{oblv}_{\text{D-mod}(Z_1)}"'] \\
\text{IndCoh}(Z_2) \arrow[u, "f^!"] & \text{D-mod}(Z_2). \arrow[l, "\mathbf{oblv}_{\text{D-mod}(Z_2)}"'] \arrow[u, "f^!"']
\end{tikzcd}$$

In particular, by taking $Z_1 = Z$ and $Z_2 = \text{pt}$, we obtain that the dualizing complex ω_Z, initially defined as an object of $\text{IndCoh}(Z)$, naturally upgrades to (i.e., is the image under $\mathbf{oblv}_{\text{D-mod}(Z)}$ of) a canonically defined object of $\text{D-mod}(Z)$. By a slight abuse of notation, we denote the latter by the same character ω_Z.

As a consequence of Lemma 3.2.2 and the conservativeness of the functor $\mathbf{oblv}_{\text{D-mod}(Z)}$, we obtain:

Lemma 5.1.6. *Let a morphism $f : Z_1 \to Z_2$ be surjective on k-points. Then the functor $f^! : \text{D-mod}(Z_2) \to \text{D-mod}(Z_1)$ is conservative.*

5.1.7. *Tensor product.* For a pair of DG schemes Z_1 and Z_2 we have a canonical (continuous) functor

$$\text{D-mod}(Z_1) \otimes \text{D-mod}(Z_2) \to \text{D-mod}(Z_1 \times Z_2),$$

which is an equivalence if Z_1 and Z_2 are quasi-compact.

Remark 5.1.8. According to Corollary 8.3.4 below, quasi-compactness of *one* of the DG schemes is enough.

In particular, we have a functor of tensor product

$$\text{D-mod}(Z) \otimes \text{D-mod}(Z) \to \text{D-mod}(Z)$$

equal to

$$\text{D-mod}(Z) \otimes \text{D-mod}(Z) \to \text{D-mod}(Z \times Z) \xrightarrow{\Delta_Z^!} \text{D-mod}(Z).$$

We denote this functor by

$$\mathcal{M}_1, \mathcal{M}_2 \mapsto \mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2.$$

This defines a symmetric monoidal structure on the category D-mod(Z). The unit in the category is ω_Z.

By Sect. 5.1.5, we have:

$$\mathbf{oblv}_{\text{D-mod}(Z)}(\mathcal{M}_1) \overset{!}{\otimes} \mathbf{oblv}_{\text{D-mod}(Z)}(\mathcal{M}_2) \simeq \mathbf{oblv}_{\text{D-mod}(Z)}(\mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2),$$

where

$$\overset{!}{\otimes} : \mathrm{IndCoh}(Z) \otimes \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(Z)$$

is as in Sect. 3.2.3.

By adjunction, for $\mathcal{F} \in \mathrm{IndCoh}(Z)$ and $\mathcal{M} \in \text{D-mod}(Z)$, we have a canonical map

$$\mathbf{ind}_{\text{D-mod}(Z)} \left(\mathcal{F} \overset{!}{\otimes} \mathbf{oblv}_{\text{D-mod}(Z)}(\mathcal{M}) \right) \to \mathbf{ind}_{\text{D-mod}(Z)}(\mathcal{F}) \overset{!}{\otimes} \mathcal{M}. \tag{5.1}$$

It is easy to show (e.g., using Kashiwara's lemma below) that the map (5.1) is an isomorphism.

5.1.9. *Kashiwara's lemma.* If $i : Z_1 \to Z_2$ is a closed embedding, [18] then the functor $i_{\mathrm{dR},*}$ induces an equivalence

$$\text{D-mod}(Z_1) \to \text{D-mod}(Z_2)_{Z_1}, \tag{5.2}$$

where $\text{D-mod}(Z_2)_{Z_1}$ is the full subcategory of $\text{D-mod}(Z_2)$ that consists of objects that vanish on the complement $Z_2 - Z_1$. The inverse equivalence is given by $i^!|_{\text{D-mod}(Z_2)_{Z_1}}$.

This observation allows to reduce the local aspects of the theory of D-modules on DG schemes to those on smooth classical schemes.

5.1.10. *Topological invariance.* In particular, if a map $i : Z_1 \to Z_2$ is such that the induced map

$$({}^{cl}Z_1)_{red} \to ({}^{cl}Z_2)_{red}$$

is an isomorphism, then the functors

$$i_{\mathrm{dR},*} : \text{D-mod}(Z_1) \rightleftarrows \text{D-mod}(Z_2) : i^! \tag{5.3}$$

are equivalences.

This shows, in particular, that for any Z, pullback along the canonical map $({}^{cl}Z)_{red} \to Z$ induces an equivalence

$$\text{D-mod}(Z) \to \text{D-mod}(({}^{cl}Z)_{red}).$$

So, when discussing the aspects of the theory of D-modules that do not involve the functors $\mathbf{ind}_{\text{D-mod}(Z)}$ and $\mathbf{oblv}_{\text{D-mod}(Z)}$, we can (and will) restrict ourselves to classical schemes, and even assume that they are reduced, without losing in generality.

[18] A map of DG schemes is called a closed embedding if the map of the underlying classical schemes is.

5.1.11. *t-structure.* The category $\text{D-mod}(Z)$ has a canonical t-structure. It is defined so that $\text{D-mod}(Z)^{>0}$ consists of all $\mathcal{F} \in \text{D-mod}(Z)$ such that $\mathbf{oblv}_{\text{D-mod}(Z)}(\mathcal{F}) \in \text{IndCoh}(Z)^{>0}$.

For a closed embedding $i : Z_1 \to Z_2$, the functor $i_{\text{dR},*}$ is t-exact. In particular, the equivalence (5.2) is compactible with t-structures, where the t-structure on $\text{D-mod}(Z_2)_{Z_1}$ is induced by that on $\text{D-mod}(Z_2)$.

By definition, the functor $\mathbf{oblv}_{\text{D-mod}(Z)}$ is left t-exact. If Z is smooth, then $\mathbf{oblv}_{\text{D-mod}(Z)}$ is t-exact. For any quasi-compact Z it has finite cohomological amplitude: to prove this, reduce to the case where Z is affine and then embed Z into a smooth classical scheme.

For the same reason, the functor $\mathbf{ind}_{\text{D-mod}(Z)}$ is always t-exact.

Lemma 5.1.12. *The t-structure on* $\text{D-mod}(Z)$ *is left-complete and is compatible with filtered colimits.*

The meaning of these words is explained in Lemma 1.2.8.

Proof. Compatibility with filtered colimits is clear from the definition of $\text{D-mod}(Z)^{>0}$. To prove left-completeness, it suffices to consider the case where Z is affine. In this case it follows from the existence of a conservative t-exact functor $\Phi : \text{D-mod}(Z) \to \text{Vect}$ commuting with limits. To construct such Φ, choose an embedding $i : Z \hookrightarrow Y$ with Y affine and smooth, then take Φ to be the composition of $i_{\text{dR},*} : \text{D-mod}(Z) \to \text{D-mod}(Y)$, $\mathbf{oblv}_{\text{D-mod}(Y)} : \text{D-mod}(Y) \to \text{IndCoh}(Y)$, $\Psi_{\mathcal{Y}} : \text{IndCoh}(Y) \simeq \text{QCoh}(Y)$ and $\Gamma : \text{QCoh}(Y) \to \text{Vect}$. $\square$

5.1.13. *Relation between* $\text{D-mod}(Z)$ *and* $\text{QCoh}(Z)$*.* It follows from Lemma 5.1.12 that the functor

$$\mathbf{ind}_{\text{D-mod}(Z)} : \text{IndCoh}(Z) \to \text{D-mod}(Z)$$

canonically factors as

$$\text{IndCoh}(Z) \xrightarrow{\Psi_Z} \text{QCoh}(Z) \to \text{D-mod}(Z).$$

This is a formal consequence of the fact that the functor Ψ_Z identifies $\text{QCoh}(Z)$ with the left completion of $\text{IndCoh}(Z)$ with respect to its t-structure, while $\text{D-mod}(Z)$ is left-complete and $\mathbf{ind}_{\text{D-mod}(Z)}$ is right t-exact.

We shall denote the resulting functor $\text{QCoh}(Z) \to \text{D-mod}(Z)$ by $'\mathbf{ind}_{\text{D-mod}(Z)}$. In addition, we have a functor $'\mathbf{oblv}_{\text{D-mod}(Z)} : \text{D-mod}(Z) \to \text{QCoh}(Z)$ defined as

$$'\mathbf{oblv}_{\text{D-mod}(Z)} := \Psi_Z \circ \mathbf{oblv}_{\text{D-mod}(Z)}.$$

It follows from Kashiwara's lemma that the functor $'\mathbf{oblv}_{\text{D-mod}(Z)}$ is also conservative.

Assume now that Z is eventually coconnective (we remind that this implies that the functor Ψ_Z admits a fully faithful 1 adjoint). Again, it follows formally that in this case the functors

$$'\mathbf{ind}_{\text{D-mod}(Z)} : \text{QCoh}(Z) \rightleftarrows \text{D-mod}(Z) : {'\mathbf{oblv}_{\text{D-mod}(Z)}}$$

are mutually adjoint.

Remark 5.1.14. We emphasize, however, that the latter case is *false* of Z is not essentially coconnective. E.g., in the latter case the functor $'\mathbf{ind}_{\text{D-mod}(Z)}$ does not send compact objects to compact ones.

Remark 5.1.15. The category $\text{D-mod}(Z)$, equipped with the functor $'\mathbf{oblv}_{\text{D-mod}(Z)}$, is the more familiar realization of D-modules as right D-modules (but which only works in the eventually coconnective case).

5.1.16. *The "left" realization.* For completeness let us mention that in addition to $\mathbf{oblv}_{\text{D-mod}(Z)}$, there is another canonically defined forgetful functor

$$\mathbf{oblv}^{\text{left}}_{\text{D-mod}(Z)} : \text{D-mod}(Z) \to \text{QCoh}(Z),$$

responsible for the realization of D-mod(Z) as "left D-modules".

For a map $f : Z_1 \to Z_2$ of DG schemes, the following diagram naturally commutes:

$$\begin{CD} \text{QCoh}(Z_1) @<{\mathbf{oblv}^{\text{left}}_{\text{D-mod}(Z_1)}}<< \text{D-mod}(Z_1) \\ @A{f^*}AA @AA{f^!}A \\ \text{QCoh}(Z_2) @<{\mathbf{oblv}^{\text{left}}_{\text{D-mod}(Z_2)}}<< \text{D-mod}(Z_2). \end{CD}$$

The functors $\mathbf{oblv}^{\text{left}}_{\text{D-mod}(Z)}$ and $\mathbf{oblv}_{\text{D-mod}(Z)}$ are related by the formula

$$\mathbf{oblv}_{\text{D-mod}(Z)}(\mathcal{M}) \simeq \mathbf{oblv}^{\text{left}}_{\text{D-mod}(Z)}(\mathcal{M}) \otimes \omega_Z,$$

where $\otimes$ is understood in the sense of the action of QCoh(Z) on IndCoh(Z), see Sect. 3.2.1.

In addition, we have a functor

$$\mathbf{ind}^{\text{left}}_{\text{D-mod}(Z)} : \text{QCoh}(Z) \to \text{D-mod}(Z)$$

defined by the formula

$$\mathbf{ind}^{\text{left}}_{\text{D-mod}(Z)}(\mathcal{F}) := \mathbf{ind}_{\text{D-mod}(Z)}(\mathcal{F} \otimes \omega_Z).$$

It again follows formally that when Z is eventually coconnective, the functors

$$(\mathbf{ind}^{\text{left}}_{\text{D-mod}(Z)}, \mathbf{oblv}^{\text{left}}_{\text{D-mod}(Z)})$$

form an adjoint pair.

For any Z one has

$$'\mathbf{oblv}_{\text{D-mod}(Z)}(\mathcal{M}) \simeq \mathbf{oblv}^{\text{left}}_{\text{D-mod}(Z)}(\mathcal{M}) \otimes \Psi_Z(\omega_Z), \quad \mathcal{M} \in \text{D-mod}(Z) \tag{5.4}$$

and

$$\mathbf{ind}^{\text{left}}_{\text{D-mod}(Z)}(\mathcal{F}) \simeq {}'\mathbf{ind}_{\text{D-mod}(Z)}(\mathcal{F} \otimes \Psi_Z(\omega_Z)), \quad \mathcal{F} \in \text{QCoh}(Z), \tag{5.5}$$

where $\Psi_Z : \text{IndCoh}(Z) \to \text{QCoh}(Z)$ is the functor of Sect. 3.2.4.

5.1.17. *Coherence and compact generation.* Let D-mod$_{\text{coh}}(Z) \subset$ D-mod(Z) denote the full subcategory of bounded complexes whose cohomology sheaves are coherent (i.e., locally finitely generated) D-modules.

If Z is quasi-compact, we have D-mod$_{\text{coh}}(Z) =$ D-mod$(Z)^c$, and this subcategory generates D-mod(Z). I.e.,

$$\text{D-mod}(Z) \simeq \text{Ind}(\text{D-mod}_{\text{coh}}(Z)).$$

In fact, this is a formal consequence of the following three facts: (a) that the functor $\mathbf{oblv}_{\text{D-mod}(Z)}$ is conservative; (b) that $\mathbf{ind}_{\text{D-mod}(Z)}$ sends Coh(Z) to D-mod$_{\text{coh}}(Z)$ (which follows from Kashiwara's lemma), and (c) that for Z quasi-compact Coh(Z) compactly generates IndCoh(Z).

5.2. **The de Rham cohomology functor on DG schemes.**

5.2.1. Let $f : Z_1 \to Z_2$ be a quasi-compact morphism between DG schemes. In this case the classical theory of D-modules constructs a continuous functor:

$$f_{\mathrm{dR},*} : \text{D-mod}(Z_1) \to \text{D-mod}(Z_2).$$

The following are the some of the key features of this functor:

(i) The assignment $f \rightsquigarrow f_{\mathrm{dR},*}$ is compatible with composition of functors in the natural sense.

(ii) For f proper, the functor $f_{\mathrm{dR},*}$ is the left adjoint to $f^!$.

(iii) For f an open embedding, the functor $f_{\mathrm{dR},*}$ is the right adjoint to $f^!$.

(iv) For a Cartesian square

$$\begin{tikzcd} Z_1' \arrow[r, "g_1"] \arrow[d, "f'"'] & Z_1 \arrow[d, "f"] \\ Z_2' \arrow[r, "g_2"] & Z_2 \end{tikzcd}$$

we have a canonical isomorphism of functors $\text{D-mod}(Z_1) \to \text{D-mod}(Z_2')$

$$f'_{\mathrm{dR},*} \circ g_1^! \simeq g_2^! \circ f_{\mathrm{dR},*}. \tag{5.6}$$

However, even the formulation of these properties in the framework on ∞-categories is not straightforward. For example, it is not so easy to formulate the compatibility between the isomorphisms (i) and (iv), and also between (ii) or (iii) and (iv). [19]

At the same time, an ∞-category formulation is necessary for the treatment of the category of D-modules on stacks, as the latter involves taking limits in $\mathrm{DGCat}_{\mathrm{cont}}$.

5.2.2. We shall adopt the approach taken in [FG], Sect. 1.4.3, which was initially suggested by J. Lurie; it will be developed in detail in [GR2].

Namely, let $(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}$ be the $(\infty, 1)$-category whose objects are the same as those of $\mathrm{DGSch}_{\mathrm{aft}}$, and where the ∞-*groupoid* of 1-morphisms $\mathrm{Maps}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}}(Z_1, Z_2)$ is that of correspondences

$$\begin{tikzcd} & Z_{1,2} \arrow[dl, "f_l"'] \arrow[dr, "f_r"] & \\ Z_1 & & Z_2. \end{tikzcd} \tag{5.7}$$

Compositions in this category are defined by forming Cartesian products:

$$Z_{2,3} \circ Z_{1,2} = Z_{1,3} :$$

[19] Note that when f is either proper or open, there is a canonical map in one direction in (5.6) by adjunction. So, in particular, we must have a compatibility condition that says that in either of these cases, the two maps in (5.6): one arising by adjunction and the other by the data of (iv), must coincide.

$$
\begin{tikzcd}
 & & Z_{1,3} \arrow[dl] \arrow[dr] & & \\
 & Z_{1,2} \arrow[dl] \arrow[dr] & & Z_{2,3} \arrow[dl] \arrow[dr] & \\
Z_1 & & Z_2 & & Z_3.
\end{tikzcd}
\tag{5.8}
$$

5.2.3. The category $(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}$ contains $\mathrm{DGSch}_{\mathrm{aft}}$ and $(\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}}$ as 1-full subcategories where we restrict 1-morphisms by requiring that f_l (resp., f_r) be an isomorphism.

The theory of D-modules is a functor

$$\text{D-mod}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}} : (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}} \to \mathrm{DGCat}_{\mathrm{cont}} .$$

At the level of objects, this functor assigns to $Z \in \mathrm{DGSch}_{\mathrm{aft}}$ the category $\text{D-mod}(Z)$.

The restriction of $\text{D-mod}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}}$ to $\mathrm{DGSch}_{\mathrm{aft}} \subset (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}$, denoted $\text{D-mod}_{\mathrm{DGSch}_{\mathrm{aft}}}$, expresses our ability to take $f_{\mathrm{dR},*} : \text{D-mod}(Z_1) \to \text{D-mod}(Z_2)$, and corresponds to diagrams of the form

$$
\begin{tikzcd}
 & Z_1 \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\
Z_1 & & Z_2.
\end{tikzcd}
\tag{5.9}
$$

The restriction of $\text{D-mod}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}}$ to $(\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \subset (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}$, which we denote by $\text{D-mod}^!_{\mathrm{DGSch}_{\mathrm{aft}}}$, expresses our ability to take $f^! : \text{D-mod}(Z_2) \to \text{D-mod}(Z_1)$, and corresponds to diagrams of the form

$$
\begin{tikzcd}
 & Z_2 \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\
Z_1 & & Z_2.
\end{tikzcd}
\tag{5.10}
$$

The base change isomorphism of Sect. 5.2.1(iv) is encoded by the functoriality of D-mod.

As is explained in [FG], Sects. 1.4.5 and 1.4.6, the datum of the functor $\text{D-mod}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}}$ also contains the data of adjunction for $(f^!, f_{\mathrm{dR},*})$ when f is an open embedding, and for $(f_{\mathrm{dR},*}, f^!)$ when f is proper.

Unfortunately, there currently is no reference in the literature for the construction of the functor $\text{D-mod}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}}$ with the above properties. However, a construction of a similar framework for IndCoh instead of D-mod has been indicated in [IndCoh], Sects. 5 and 6.

5.2.4. An additional part of data in the functor $\text{D-mod}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}}$ is the following one:

The functor $\text{D-mod}^!_{\mathrm{DGSch}_{\mathrm{aft}}} : (\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$ comes equipped with a natural transformation

$$\mathbf{oblv}_{\text{D-mod}} : \text{D-mod}^!_{\mathrm{DGSch}_{\mathrm{aft}}} \to \mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}},$$

where

$$\mathrm{IndCoh}^!_{\mathrm{DGSch}_{\mathrm{aft}}} : (\mathrm{DGSch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is the functor of Sect. 3.2.1. [20]

The functor $\text{D-mod}_{\mathrm{DGSch}_{\mathrm{aft}}} : \mathrm{DGSch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}$ comes equipped with a natural transformation

$$\mathbf{ind}_{\text{D-mod}} : \mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{aft}}} \to \text{D-mod}_{\mathrm{DGSch}_{\mathrm{aft}}},$$

where

$$\mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{aft}}} : \mathrm{DGSch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is the functor of [IndCoh, Sect. 5.6.1]. In particular, for a morphism $f : Z_1 \to Z_2$ of quasi-compact schemes, we have a commutative diagram

$$\begin{tikzcd} \text{D-mod}(Z_1) \arrow[d, "f_{\mathrm{dR},*}"'] & \mathrm{IndCoh}(Z_1) \arrow[l, "\mathbf{ind}_{\text{D-mod}(Z_1)}"'] \arrow[d, "f_*^{\mathrm{IndCoh}}"] \\ \text{D-mod}(Z_1) & \mathrm{IndCoh}(Z_1). \arrow[l, "\mathbf{ind}_{\text{D-mod}(Z_1)}"'] \end{tikzcd} \tag{5.11}$$

Remark 5.2.5. In principle, one would like to formulate the compatibility of the entire datum of the functor $\text{D-mod}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr}}}$ with that of the functor $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}}$ of [IndCoh, Sect. 5.6.1]. However, we cannot do this while staying in the world of $(\infty, 1)$-categories, as some of the natural transformations involved are not isomorphisms.

5.2.6. *Projection formula.* As in (3.4), from (5.6) one obtains that f satisfies projection formula for D-modules: for a map $f : Z_1 \to Z_2$ of quasi-compact DG schemes, and $\mathcal{M}_i \in \text{D-mod}(Z_i)$, $i = 1, 2$ we have a canonical isomorphism

$$\mathcal{M}_2 \otimes f_{\mathrm{dR},*}(\mathcal{M}_1) \simeq f_{\mathrm{dR},*}(f^!(\mathcal{M}_2) \otimes \mathcal{M}_1), \tag{5.12}$$

functorial in $\mathcal{M}_i$.

5.2.7. *De Rham cohomology.* For $Z \in \mathrm{DGSch}_{\mathrm{aft}}$ we obtain a functor

$$\Gamma_{\mathrm{dR}}(Z, -) := (p_Z)_{\mathrm{dR},*} : \text{D-mod}(Z) \to \mathrm{Vect},$$

where $p_Z : Z \to \mathrm{pt}$.

This functor is co-representable by an object $k_Z \in \text{D-mod}(Z)$, i.e.,

$$\Gamma_{\mathrm{dR}}(Z, \mathcal{M}) = \mathcal{M}aps(k_Z, \mathcal{M}). \tag{5.13}$$

As Z was assumed quasi-compact, the functor $\Gamma_{\mathrm{dR}}(Z, -)$ is continuous, so $k_Z \in \text{D-mod}(Z)$ is compact.

Remark 5.2.8. By Sect. 5.1.4, the object $k_Z \in \text{D-mod}(Z)$ is defined for any Z, not necessarily quasi-compact. However, in general, it will fail to be compact as an object of $\text{D-mod}(Z)$.

[20]For an individual morphism, this datum is the one in Sect. 5.1.5.

5.2.9. Let $f : Z_1 \to Z_2$ be a map between quasi-compact schemes. Since

$$\Gamma_{\mathrm{dR}}(Z_1, -) \simeq \Gamma_{\mathrm{dR}}(Z_2, f_{\mathrm{dR},*}(-)),$$

we obtain that the *partially defined* left adjoint f^*_{dR} to $f_{\mathrm{dR},*}$ is defined on k_{Z_2}, and we have a canonical isomorphism

$$f^*_{\mathrm{dR}}(k_{Z_2}) \simeq k_{Z_1}.$$

5.3. **Verdier duality on DG schemes.**

5.3.1. For a DG scheme Z, there is a (unique) involutive anti self-equivalence

$$\mathbb{D}_Z^{\mathrm{Verdier}} : (\text{D-mod}_{\mathrm{coh}}(Z))^{\mathrm{op}} \xrightarrow{\sim} \text{D-mod}_{\mathrm{coh}}(Z)$$

(called Verdier duality) such that

$$\mathcal{M}aps_{\text{D-mod}(Z)}(\mathbb{D}_Z^{\mathrm{Verdier}}(\mathcal{M}), \mathcal{M}') \simeq \Gamma_{\mathrm{dR}}(Z, \mathcal{M} \overset{!}{\otimes} \mathcal{M}'), \tag{5.14}$$

for $\mathcal{M} \in \text{D-mod}_{\mathrm{coh}}(Z)$, $\mathcal{M}' \in \text{D-mod}(Z)$.

Let ω_Z and k_Z be as in Sect. 5.1.5 and 5.2.7. Then $\omega_Z, k_Z \in \text{D-mod}_{\mathrm{coh}}(Z)$ and

$$k_Z \simeq \mathbb{D}_Z^{\mathrm{Verdier}}(\omega_Z).$$

5.3.2. *Verdier and Serre duality.* If $\mathcal{F} \in \mathrm{Coh}(Z)$ then $\mathbf{ind}_{\text{D-mod}(Z)}(\mathcal{F}) \in \text{D-mod}_{\mathrm{coh}}(Z)$. We now claim:

Lemma 5.3.3. *There exists a canonical isomorphism*

$$\mathbb{D}_Z^{\mathrm{Verdier}}\left(\mathbf{ind}_{\text{D-mod}(Z)}(\mathcal{F})\right) \simeq \mathbf{ind}_{\text{D-mod}(Z)}\left(\mathbb{D}_Z^{\mathrm{Serre}}(\mathcal{F})\right). \tag{5.15}$$

Proof. Follows by combining isomorphsms (5.14), (5.11) and (5.1), and Proposition 4.4.4 (for DG schemes). ☐

5.3.4. *Ind-extending Verdier duality.* For Z quasi-compact, ind-extending Verdier duality, by Sect. 4.1.3(ii'), we obtain an identification

$$\mathbf{D}_Z^{\mathrm{Verdier}} : \text{D-mod}(Z)^\vee \simeq \text{D-mod}(Z), \tag{5.16}$$

where $\text{D-mod}(Z)^\vee$ is the dual DG category (see Sect. 4.1.1).

By (5.14), the corresponding pairing

$$\epsilon_{\text{D-mod}(Z)} : \text{D-mod}(Z) \otimes \text{D-mod}(Z) \to \mathrm{Vect}$$

equals the composition

$$\text{D-mod}(Z) \otimes \text{D-mod}(Z) \to \text{D-mod}(Z \times Z) \xrightarrow{\Delta_Z^!} \text{D-mod}(Z) \xrightarrow{\Gamma_{\mathrm{dR}}(Z,-)} \mathrm{Vect}.$$

As in the proof of Proposition 4.4.9, the base change isomorphism implies that the co-evaluation functor

$$\mu_{\text{D-mod}(Z)} : \mathrm{Vect} \to \text{D-mod}(Z) \otimes \text{D-mod}(Z)$$

is given by

$$\mathrm{Vect} \xrightarrow{\omega_Z \otimes -} \text{D-mod}(Z) \xrightarrow{\Delta_{\mathrm{dR},*}} \text{D-mod}(Z \times Z) \simeq \text{D-mod}(Z) \otimes \text{D-mod}(Z).$$

The latter implies, in turn, that for a map of quasi-compact DG schemes $f : Z_1 \to Z_2$, under the identifications

$$\mathbf{D}_{Z_i}^{\mathrm{Verdier}} : \text{D-mod}(Z_i)^\vee \simeq \text{D-mod}(Z_i),$$

we have:

$$(f^!)^\vee \simeq f_{\mathrm{dR},*} \tag{5.17}$$

(see Sect. 4.1.4 for the notion of dual functor).

Note also that by [GL:DG, Lemma 2.3.3], the isomorphism (5.15) can also be formulated as saying that

$$(\mathbf{oblv}_{\text{D-mod}(Z)})^\vee \simeq \mathbf{ind}_{\text{D-mod}(Z)}$$

with respect to the identifications

$$\mathbf{D}_Z^{\mathrm{Verdier}} : \text{D-mod}(Z)^\vee \simeq \text{D-mod}(Z) \text{ and } \mathbf{D}_Z^{\mathrm{Serre}} : (\mathrm{IndCoh}(Z))^\vee \simeq \mathrm{IndCoh}(Z),$$

given by Verdier and Serre dualities, respectively.

5.3.5. *Smooth pullbacks.* If f is smooth then the functor $f_{\mathrm{dR},*}$ admits a left adjoint, which we denote by f^*_{dR}. Being a left adjoint, the functor f^*_{dR} is continuous. If f is of constant relative dimension n, we have a canonical isomorphism

$$f^*_{\mathrm{dR}} \simeq f^![-2n]. \tag{5.18}$$

One has

$$f^*_{\mathrm{dR}}(\text{D-mod}_{\mathrm{coh}}(Z_2)) \subset \text{D-mod}_{\mathrm{coh}}(Z_1), \quad f^!(\text{D-mod}_{\mathrm{coh}}(Z_2)) \subset \text{D-mod}_{\mathrm{coh}}(Z_1), \tag{5.19}$$

$$\mathbb{D}^{\mathrm{Verdier}}_{Z_1}(f^*_{\mathrm{dR}}(\mathcal{M})) \simeq f^!\left(\mathbb{D}^{\mathrm{Verdier}}_{Z_2}(\mathcal{M})\right), \quad \mathcal{M} \in \text{D-mod}_{\mathrm{coh}}(Z_2). \tag{5.20}$$

Remark 5.3.6. Assume that Z_1 and Z_2 are quasi-compact (which we can, as the above assertions are Zariski-local). Recall that in this case $\text{D-mod}(Z_i)^c = \text{D-mod}_{\mathrm{coh}}(Z_i)$. We obtain that (5.19) follows from the fact that f^*_{dR} preserves compactness (because it has a continuous right adjoint), and (5.20) follows from formula (5.17) combined with [GL:DG, Lemma 2.3.3].

5.3.7. For $\mathcal{M}', \mathcal{M}'' \in \text{D-mod}(Z_2)$ by adjunction and the projection formula (5.12) we obtain a map

$$f^*_{\mathrm{dR}}(\mathcal{M}' \overset{!}{\otimes} \mathcal{M}'') \to f^*_{\mathrm{dR}}(\mathcal{M}') \overset{!}{\otimes} f^!(\mathcal{M}''). \tag{5.21}$$

However, it easily follows from (5.18) that (5.21) is an isomorphism.

6. D-modules on stacks

In this section we review the theory of D-modules on algebraic stacks to be used later in the paper. On the one hand, this theory is well-known, at least at the level of triangulated categories. However, as we could not find a single source that contains all the relevant facts, we decided to include the present section for the reader's convenience.

6.1. **D-modules on prestacks.**

6.1.1. Let $\mathcal{Y}$ be a prestack. The category D-mod$(\mathcal{Y})$ is defined as

$$\underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \text{D-mod}(S), \tag{6.1}$$

where we view the assignment

$$(S,g) \rightsquigarrow \text{D-mod}(S)$$

as a functor between ∞-categories

$$((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

obtained by restriction under the forgetful map $(\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}} \to \mathrm{DGSch}_{\mathrm{aft}}$ of the functor

$$\text{D-mod}^!_{\mathrm{DGSch}_{\mathrm{aft}}} : \mathrm{DGSch}^{\mathrm{op}}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

Concretely, an object $\mathcal{M}$ of D-mod$(\mathcal{Y})$ is an assignment for every $g : S \to \mathcal{Y}$ of an object $g^!(\mathcal{M}) \in \text{D-mod}(S)$, and a homotopy-coherent system of isomorphisms

$$f^!(g^!(\mathcal{M})) \simeq (g\circ f)^!(\mathcal{M}) \in \text{D-mod}(S')$$

for maps of DG schemes $f : S' \to S$.

In the above limit one can replace the category of quasi-compact DG schemes by its subcategory of affine DG schemes, or by a larger category of all DG schemes; this is due to the Zariski descent property of D-modules, see Sect. 5.1.4.

In addition, using the fppf descent property for D-modules (see Sect. 5.1.4), we can replace the categories $(\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}}$ (resp., $(\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}}$) by any of the indexing categories A as in Sect. 1.2.5. The proof follows from [IndCoh, Corollary 11.2.4].

6.1.2. According to [GR1, Corollary 2.3.9], we can equivalently define D-mod$(\mathcal{Y})$ as

$$\mathrm{IndCoh}(\mathcal{Y}_{\mathrm{dR}}),$$

where $\mathcal{Y}_{\mathrm{dR}}$ is as an [GR1, Sect. 1.1.1].

6.1.3. Tautologically, for a morphism $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ between prestacks, we have a functor

$$\pi^! : \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1).$$

In particular, for any prestack $\mathcal{Y}$, there exists a canonically defined object

$$\omega_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y})$$

equal to $(p_{\mathcal{Y}})^!(k)$ for $p_{\mathcal{Y}} : \mathcal{Y} \to \mathrm{pt}$.

6.1.4. It follows from Sect. 5.1.10 that if a morphism of prestacks $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ induces an isomorphism of the underlying classical prestacks ${}^{cl}\mathcal{Y}_1 \to {}^{cl}\mathcal{Y}_2$, then the functor

$$\pi^! : \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1)$$

is an equivalence.

So, for a prestack $\mathcal{Y}$, the category D-mod$(\mathcal{Y})$ only depends on the underlying classical prestack.

6.1.5. Just as in Sect. 5.1.7, for a pair of prestacks $\mathcal{Y}_1$ and $\mathcal{Y}_2$ one has a canonical (continuous) functor

$$\text{D-mod}(\mathcal{Y}_1) \otimes \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

and a functor of tensor product

$$\text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Y})$$

defined as the composition

$$\text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Y} \times \mathcal{Y}) \xrightarrow{\Delta^!_{\mathcal{Y}}} \text{D-mod}(\mathcal{Y}).$$

6.1.6. The natural transformation $\mathbf{oblv}_{\text{D-mod}}$ of Sect. 5.2.4 gives rise to a continuous *conservative* functor

$$\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})} : \text{D-mod}(\mathcal{Y}) \to \text{IndCoh}(\mathcal{Y}),$$

which is compatible with morphisms of prestacks under !-pullbacks.

As in the case of DG schemes, we can interpret the functor $\mathbf{oblv}_{\text{D-mod}}$ as pullback along the tautological morphism of prestacks $\mathcal{Y} \to \mathcal{Y}_{\text{dR}}$.

However, it is not clear, and most probably not true, that for a general prestack this functor admits a left adjoint. Neither is it possible for a general prestack $\mathcal{Y}$ to consider the functor $'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}$ (because the functor $\Psi_{\mathcal{Y}}$ is a feature of DG schemes or algebraic stacks).

In addition, the functor $\mathbf{oblv}^{\text{left}}_{\text{D-mod}(-)}$ for schemes mentioned in Sect. 5.1.16 gives rise to a functor

$$\mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathcal{Y})} : \text{D-mod}(\mathcal{Y}) \to \text{QCoh}(\mathcal{Y}),$$

which is compatible with morphisms of prestacks under !-pullbacks on D-mod and usual *-pullbacks on QCoh.

6.1.7. *Quasi-compact schematic morphisms.* Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a schematic and quasi-compact morphism between prestacks. The functor of $(\text{dR}, *)$-pushforward for DG schemes gives rise to a continuous functor

$$\pi_{\text{dR},*} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2).$$

As in the case of the $(\text{IndCoh}, *)$-pushforward, one constructs the functor $\pi_{\text{dR},*}$ as follows:

For $(S_2, g_2) \in (\text{DGSch}_{\text{aft}})_{/\mathcal{Y}}$, we set

$$g_2^!(\pi_{\text{dR},*}(-)) := (\pi_S)_{\text{dR},*} \circ g_1^!(-)$$

for the morphisms in the Cartesian diagram

$$\begin{tikzcd} S_1 \arrow[r, "g_1"] \arrow[d, "\pi_S"'] & \mathcal{Y}_1 \arrow[d, "\pi"] \\ S_2 \arrow[r, "g_2"] & \mathcal{Y}_2. \end{tikzcd}$$

The data of compatibility of the assignment

$$(S_2, g_2) \rightsquigarrow (\pi_S)_{\text{dR},*} \circ g_1^!(-)$$

under !-pullbacks for maps in $(\text{DGSch}_{\text{aft}})_{/\mathcal{Y}}$ is given by base change isomorphisms (5.6).

Moreover, the formation $\pi_{\mathrm{dR},*}$ is also endowed with base change isomorphisms with respect to !-pullbacks for Cartesian squares of prestacks

$$\begin{tikzcd} \mathcal{Y}_1' \arrow[r] \arrow[d, "\pi'"'] & \mathcal{Y}_1 \arrow[d, "\pi"] \\ \mathcal{Y}_2' \arrow[r] & \mathcal{Y}_2, \end{tikzcd}$$

where the vertical maps are schematic and quasi-compact.

By construction, the projection formula for morphisms between quasi-compact schemes, i.e., (5.12), implies one for π. That is, we have a functorial isomorphism

$$\mathcal{M}_2 \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{M}_1) \simeq \pi_{\mathrm{dR},*}(\pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1), \quad \mathcal{M}_i \in \text{D-mod}(\mathcal{Y}_i). \tag{6.2}$$

Remark 6.1.8. We emphasize again that the isomorphisms neither in base change nor in projection formula arise by adjunction from a priori existing maps.

6.1.9. Let $\mathcal{Y}_i$ be prestacks, and let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism which is k-representable for some k. As in the case of IndCoh, we will only need the cases of either π being schematic, or 1-representable. Assume also that π is smooth.

In this case we also have a naturally defined functor

$$\pi^*_{\mathrm{dR}} : \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1).$$

By (5.18), if π is of relative dimension n, we have a canonical isomorphism

$$\pi^*_{\mathrm{dR}} \simeq \pi^![-2n].$$

For $\mathcal{M}', \mathcal{M}'' \in \text{D-mod}(\mathcal{Y}_2)$, from (5.21) we obtain a canonical isomorphism

$$\pi^*_{\mathrm{dR}}(\mathcal{M}' \overset{!}{\otimes} \mathcal{M}'') \to \pi^*_{\mathrm{dR}}(\mathcal{M}') \overset{!}{\otimes} \pi^!(\mathcal{M}''), \tag{6.3}$$

Finally, assuming that π is, in addition, schematic and quasi-compact, we obtain that the functors $(\pi^*_{\mathrm{dR}}, \pi_{\mathrm{dR},*})$ are naturally adjoint.

6.2. **D-modules on algebraic stacks.** From now until the end of this section we shall assume that $\mathcal{Y}$ is an algebraic stack.

6.2.1. As was mentioned above, in the formation of the limit (6.1), one can replace the category $(\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}}$ by $\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}$.

I.e., the functor

$$\text{D-mod}(\mathcal{Y}) \simeq \underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}})^{\mathrm{op}}}{\underleftarrow{lim}} \text{D-mod}(S) \to \underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} \text{D-mod}(S), \tag{6.4}$$

obtained by restriction, is an equivalence.

6.2.2. Furthermore, using $(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}$ as indexing category, D-mod$(\mathcal{Y})$ can be also realized as the limit

$$\underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} \text{D-mod}(S), \tag{6.5}$$

which is formed with $f^*_{\mathrm{dR}} : \text{D-mod}(S') \to \text{D-mod}(S)$ as transition functors. (This follows from Sect. 5.3.5.)

In addition, choosing a smooth atlas $f : Z \to \mathcal{Y}$, we have:

$$\text{D-mod}(\mathcal{Y}) \simeq \text{Tot}\left(\text{D-mod}(Z^\bullet/\mathcal{Y})\right), \tag{6.6}$$

where the cosimplicial category is formed using either !-pullback or $(\text{dR},*)$-pullback functors along the simplicial DG scheme $Z^\bullet/\mathcal{Y}$. (The assertion for !-pullbacks follows from the smooth descent property of D-modules, and that for $(\text{dR},*)$-pullbacks from Sect. 5.3.5.)

6.2.3. For an algebraic stack $\mathcal{Y}$, the category $\text{D-mod}(\mathcal{Y})$ has a (unique) t-structure such that

$$\text{D-mod}(\mathcal{Y})^{>0} = \{\mathcal{F} \in \text{D-mod}(\mathcal{Y}) \mid \mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}) \in \text{IndCoh}(\mathcal{Y})^{>0}\}.$$

The properties of this t-structure formulated in Sect. 5.1.11 for DG schemes imply similar properties for stacks. In particular, the t-structure is left-complete and compatible with colimits.

6.3. **The induction functor.** We are going to show that for algebraic stacks, the functor

$$\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})} : \text{D-mod}(\mathcal{Y}) \to \text{IndCoh}(\mathcal{Y})$$

admits a left adjoint, denoted $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$, and establish some properties of this functor.

For the main theorems of this paper we will only need the induction functor in the case when $\mathcal{Y}$ is a classical (i.e., non-derived) algebraic stack. However, for the sake of completeness, the discussion in this subsection is applicable do derived stacks as well.

Remark 6.3.1. A more streamlined treatment will be given in [GR2], where the functor of direct image on IndCoh will be developed for morphisms such as $\mathcal{Y} \to \mathcal{Y}_{\text{dR}}$, where $\mathcal{Y}$ is an algebraic stack. The latter functor *is* the sought-for induction functor.

6.3.2. Let S be a DG scheme equipped with a smooth map $g : S \to \mathcal{Y}$. Consider the category $\text{D-mod}(S)_{\text{rel}_\mathcal{Y}}$ of relative right D-modules. By definition,

$$\text{D-mod}(S)_{\text{rel}_\mathcal{Y}} := \text{IndCoh}(S_{\text{dR}} \underset{\mathcal{Y}_{\text{dR}}}{\times} \mathcal{Y}).$$

We have the forgetful functors

$$\mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}_\mathcal{Y}}} : \text{D-mod}(S)_{\text{rel}_\mathcal{Y}} \to \text{IndCoh}(S)$$

and

$$\mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}} : \text{D-mod}(S) \to \text{D-mod}(S)_{\text{rel}_\mathcal{Y}}$$

defined as pullbacks along the tautological morphisms

$$S \to S_{\text{dR}} \underset{\mathcal{Y}_{\text{dR}}}{\times} \mathcal{Y} \text{ and } S_{\text{dR}} \underset{\mathcal{Y}_{\text{dR}}}{\times} \mathcal{Y} \to S_{\text{dR}},$$

respectively. We have:

$$\mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}_\mathcal{Y}}} \circ \mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}} \simeq \mathbf{oblv}_{\text{D-mod}(S)}.$$

It is easy to see that the functor $\mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}_\mathcal{Y}}}$ is conservative. The category $\text{D-mod}(S)_{\text{rel}_\mathcal{Y}}$ carries a t-structure characterized by the property that

$$\text{D-mod}(S)^{>0}_{\text{rel}_\mathcal{Y}} = \{\mathcal{F} \in \text{D-mod}(S)_{\text{rel}_\mathcal{Y}} \mid \mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}_\mathcal{Y}}}(\mathcal{F}) \in \text{IndCoh}(S)^{>0}\}.$$

6.3.3. For a morphism $f : S' \to S$ in $\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}$, we have a naturally defined functor

$$f^! : \text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}} \to \text{D-mod}(S')_{\mathrm{rel}_{\mathcal{Y}}},$$

which makes the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(S') & \text{D-mod}(S')_{\mathrm{rel}_{\mathcal{Y}}} \arrow[l, "\mathbf{oblv}_{\text{D-mod}(S')_{\mathrm{rel}_{\mathcal{Y}}}}"'] & \text{D-mod}(S') \arrow[l, "\mathbf{oblv}_{\text{D-mod}(S')_{\mathrm{rel}\to\mathrm{abs}}}"'] \\
\mathrm{IndCoh}(S) \arrow[u, "f^!"] & \text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}} \arrow[l, "\mathbf{oblv}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}"'] \arrow[u, "f^!"'] & \text{D-mod}(S). \arrow[l, "\mathbf{oblv}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}"'] \arrow[u, "f^!"']
\end{tikzcd}$$

commute.

The assignment $S \rightsquigarrow \text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}$ has a structure functor of ∞-categories:

$$(\text{D-mod}^!_{\mathrm{rel}_{\mathcal{Y}}})_{\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}} : (\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

which is equipped with natural transformations

$$\mathrm{IndCoh}^!_{\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}} \overset{\mathbf{oblv}_{\text{D-mod}_{\mathrm{rel}_{\mathcal{Y}}}}}{\longleftarrow} (\text{D-mod}^!_{\mathrm{rel}_{\mathcal{Y}}})_{\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}} \overset{\mathbf{oblv}_{\text{D-mod}_{\mathrm{rel}\to\mathrm{abs}}}}{\longleftarrow} \text{D-mod}^!_{\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}}$$

6.3.4. For $(S, g) \in \mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}$, pullback along the morphism

$$S_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y} \to \mathcal{Y}$$

defines a functor

$$\mathrm{IndCoh}(\mathcal{Y}) \to \text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}},$$

which by a slight abuse of notation we shall denote by $g^!$. I.e.,

$$g^! \simeq \mathbf{oblv}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}} \circ g^! : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(S).$$

Furthermore, we have a functor

$$\mathrm{IndCoh}(\mathcal{Y}) \to \underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} \text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}. \tag{6.7}$$

Lemma 6.3.5. *The functor* (6.7) *is an equivalence.*

Proof. The inverse functor to (6.7) is given by the composition

$$\underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} \text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}} \overset{\mathbf{oblv}_{\text{D-mod}_{\mathrm{rel}_{\mathcal{Y}}}}}{\longrightarrow} \underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} \mathrm{IndCoh}(S) \simeq$$
$$\simeq \mathrm{IndCoh}(\mathcal{Y}).$$

□

6.3.6. Assume for a moment that $\mathcal{Y}$ is a classical (i.e., non-derived) stack. Then for $(S, g) \in \mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}$, the DG scheme S is also classical.

In this case the category $\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}$ can be described as modules over the *Lie algebroid* $T_{S/\mathcal{Y}}$ of vector fields on S vertical with respect to the map $g : S \to \mathcal{Y}$. The following is easy:

Lemma 6.3.7. *The functor $\mathbf{oblv}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}$ admits a left adjoint; we shall denote this left adjoint by $\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}$. Furthermore,*

(a) *The composion*

$$\mathbf{oblv}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}} \circ \mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}} : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(S)$$

has a filtration indexed by non-negative integers with the i-th successive quotient isomorphic to the functor $\mathrm{Sym}^i(T_{S/\mathcal{Y}}) \otimes -$.

(b) *The functors $\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}$ and $\mathbf{oblv}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}$ are both t-exact.*

(c) *Every object $\mathcal{F} \in \text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}$ admits a resolution (the relative de Rham complex) by objects of the form*

$$\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}} \left(\mathrm{Sym}^k(T_{S/\mathcal{Y}}[1]) \otimes \mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}(\mathcal{F})\right),$$

$k \in [0, \dim.\mathrm{rel}.(S/\mathcal{Y})]$.

(d) *For $\mathcal{F} \in \mathrm{IndCoh}(S)$ and $\mathcal{M} \in \text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}$, the natural map*

$$\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}(\mathcal{F} \overset{!}{\otimes} \mathbf{oblv}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}(\mathcal{M})) \to \mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}(\mathcal{F}) \overset{!}{\otimes} \mathcal{M}$$

is an isomorphism.

We obtain:

Corollary 6.3.8. *The functor $\mathbf{oblv}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}$ admits a left adjoint; we shall denote this left adjoint by $\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}$. Furthermore,*

(a) *The functor $\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}$ is right t-exact and has cohomological amplitude bounded by* $\dim.\mathrm{rel}.(S/\mathcal{Y})$.

(b) *For $\mathcal{F} \in \text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}$ and $\mathcal{M} \in \text{D-mod}(S)$, the natural map*

$$\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}(\mathcal{F} \overset{!}{\otimes} \mathbf{oblv}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}(\mathcal{M})) \to \mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}(\mathcal{F}) \overset{!}{\otimes} \mathcal{M}$$

is an isomorphism.

Proof. By Lemma 6.3.7, it is enough to consider (and prove the existence of) the functor $\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}$ applied to objects of the form

$$\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}(\mathcal{F}), \quad \mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y}).$$

However,

$$\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}(\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}(\mathcal{F})) \simeq \mathbf{ind}_{\text{D-mod}(S)}(\mathcal{F})$$

and the assertion follows. □

Remark 6.3.9. One can show that the cohomological amplitude of $\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}$ is in fact bounded by $\max\limits_{y \in \mathcal{Y}(k)} (\dim(\mathrm{Aut}(y)))$.

Remark 6.3.10. The notion of Lie algebroid is familiar for classical schemes. Its analog in the framework of derived algebraic geometry will appear in [GR2]. Assuming this theory, the assertions of of Sect. 6.3.6 are equally applicable when $\mathcal{Y}$ is a derived algebraic stack.

6.3.11. Let $\mathcal{Y}$ be again a derived algebraic stack. We have:

Proposition 6.3.12.
(a) *The functor* $\mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}}}$ *admits a left adjoint, denoted* $\mathbf{ind}_{\text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}}}$. *Both functors* $\mathbf{ind}_{\text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}}}$ *and* $\mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}}}$ *are t-exact.*
(a') *For* $\mathcal{F} \in \text{IndCoh}(S)$ *and* $\mathcal{M} \in \text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}}$, *the natural map*

$$\mathbf{ind}_{\text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}}}(\mathcal{F} \overset{!}{\otimes} \mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}}}(\mathcal{M})) \to \mathbf{ind}_{\text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}}}(\mathcal{F}) \overset{!}{\otimes} \mathcal{M}$$

is an isomorphism.
(b) *The functor* $\mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}}$ *admits a left adjoint,* $\mathbf{ind}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}}$. *The functor* $\mathbf{ind}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}}$ *is right t-exact and is of cohomological amplitude bounded by* $\dim.\text{rel}.(S/\mathcal{Y})$.
(b') *For* $\mathcal{F} \in \text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}}$ *and* $\mathcal{M} \in \text{D-mod}(S)$, *the natural map*

$$\mathbf{ind}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}}(\mathcal{F} \overset{!}{\otimes} \mathbf{oblv}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}}(\mathcal{M})) \to \mathbf{ind}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}}(\mathcal{F}) \overset{!}{\otimes} \mathcal{M}$$

is an isomorphism.

Proof. Both assertions easily reduce to the case when $\mathcal{Y}$ is classical, and there they follow from Lemma 6.3.7 and Corollary 6.3.8, respectively. □

Remark 6.3.13. In terms of the formalism that willbe explained in [GR2], the functors

$$\mathbf{ind}_{\text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}}} \text{ and } \mathbf{ind}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}}$$

correspond to the direct image along the morphisms

$$S \to S_{\text{dR}} \underset{\mathcal{Y}_{\text{dR}}}{\times} \mathcal{Y} \text{ and } S_{\text{dR}} \underset{\mathcal{Y}_{\text{dR}}}{\times} \mathcal{Y} \to S_{\text{dR}},$$

respectively.

6.3.14. Let $f : S' \to S$ be again a morphism in $\text{DGSch}_{/\mathcal{Y},\text{smooth}}$). By adjunction, the diagram

$$\begin{tikzcd} \text{D-mod}(S')_{\text{rel}_{\mathcal{Y}}} \arrow[r, "\mathbf{ind}_{\text{D-mod}(S')_{\text{rel}\to\text{abs}}}"] & \text{D-mod}(S') \\ \text{D-mod}(S)_{\text{rel}_{\mathcal{Y}}} \arrow[u, "f^!"] \arrow[r, "\mathbf{ind}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}}"] & \text{D-mod}(S) \arrow[u, "f^!"'] \end{tikzcd} \tag{6.8}$$

commutes up to a natural transformation.

Lemma 6.3.15. *The diagram* (6.8) *commutes (i.e., the natural transformation above is an isomorphism).*

Proof. The assertion reduces to the case when $\mathcal{Y}$ is classical, and there it follows from Sect. 6.3.6. □

Thus, we obtain that the functors $(S, g) \mapsto \mathbf{ind}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}}$ give rise to a natural transformation

$$(\text{D-mod}^!_{\text{rel}_{\mathcal{Y}}})_{\text{DGSch}_{/\mathcal{Y},\text{smooth}}} \overset{\mathbf{ind}_{\text{D-mod}_{\text{rel}\to\text{abs}}}}{\longrightarrow} \text{D-mod}^!_{\text{DGSch}_{/\mathcal{Y},\text{smooth}}}.$$

In particular, the assignment

$$(\mathcal{F} \in \text{IndCoh}(\mathcal{Y})) \mapsto \mathbf{ind}_{\text{D-mod}(S)_{\text{rel}\to\text{abs}}}(g^!(\mathcal{F}))$$

defines a functor

$$\text{IndCoh}(\mathcal{Y}) \to \underset{(S,g)\in(\text{DGSch}_{/\mathcal{Y},\text{smooth}})^{\text{op}}}{\underset{\longleftarrow}{lim}} \text{D-mod}(S) = \text{D-mod}(\mathcal{Y}).$$

We denote this functor by $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$.

Lemma 6.3.16. *The functor* $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ *is the left adjoint of* $\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}$.

Proof. This follows from Lemma 6.3.5. □

6.3.17. Being a left adjoint of a left t-exact functor, the functor $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ is right t-exact.

However, unlike the case of DG schemes, $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ is no longer t-exact, even when $\mathcal{Y}$ is a smooth classical stack. Rather, we have the following:

Lemma 6.3.18. *Assume that $\mathcal{Y}$ is quasi-compact. Then the functor* $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ *is of finite cohomological amplitude.*

Proof. Follows from Proposition 6.3.12(b). □

6.3.19. In the sequel we shall use the following property of the functor $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$.

First, as in the case of DG schemes, for $\mathcal{F} \in \text{IndCoh}(\mathcal{Y})$ and $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$, by adjunction we obtain a map

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}\left(\mathcal{F} \overset{!}{\otimes} \mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})\right) \to \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}) \overset{!}{\otimes} \mathcal{M}. \tag{6.9}$$

Lemma 6.3.20. *The map* (6.9) *is an isomorphism.*

Proof. Lemma 6.3.5 reduces the assertion to that of Proposition 6.3.12(b'). □

6.3.21. As in the case of DG schemes, it follows that $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ canonically factors through a functor

$$'\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} : \text{QCoh}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Y}).$$

If $\mathcal{Y}$ is eventually coconnective, it is a left adjoint of the functor

$$'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})} : \text{D-mod}(\mathcal{Y}) \to \text{QCoh}(\mathcal{Y}),$$

while the latter is conservative for any algebraic stack.

In addition, we have the functors

$$\mathbf{ind}^{\text{left}}_{\text{D-mod}(\mathcal{Y})} : \text{QCoh}(\mathcal{Y}) \rightleftarrows \text{D-mod}(\mathcal{Y}) : \mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathcal{Y})}$$

that are mutually adjoint when $\mathcal{Y}$ is eventually coconnective.

Similarly to formulas (5.4)-(5.5), for any $\mathcal{Y}$, one has

$$'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}) \simeq \mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}) \otimes \Psi_{\mathcal{Y}}(\omega_{\mathcal{Y}}), \quad \mathcal{M} \in \text{D-mod}(\mathcal{Y}) \tag{6.10}$$

and

$$\mathbf{ind}^{\text{left}}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}) \simeq {}'\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F} \otimes \Psi_{\mathcal{Y}}(\omega_{\mathcal{Y}})), \quad \mathcal{F} \in \text{QCoh}(\mathcal{Y}), \tag{6.11}$$

where $\Psi_{\mathcal{Y}}$ is as in Sect. 3.2.12.

6.4. **Example: induction for the classifying stack.** In this subsection we shall consider the case of $\mathcal{Y} = BG$, where G is an algebraic group, and

$$BG := \text{pt}/G$$

is its classifying stack. We shall describe the pair of adjoint functors

$$\mathbf{ind}_{\text{D-mod}(BG)} : \text{IndCoh}(BG) \rightleftarrows \text{D-mod}(BG) : \mathbf{oblv}_{\text{D-mod}(BG)}$$

more explicitly.

6.4.1. Take $S = \mathrm{pt}$, and let σ denote the tautological map $\mathrm{pt} \to BG$. We have a commutative diagram

$$\begin{tikzcd} \text{D-mod}(\mathrm{pt})_{\mathrm{rel}_{BG}} & \text{D-mod}(\mathrm{pt}) \arrow[l, "\mathbf{oblv}_{\text{D-mod}(\mathrm{pt})_{\mathrm{rel}\to\mathrm{abs}}}"'] \\ \mathrm{IndCoh}(BG) \arrow[u, "\sigma^!"] & \text{D-mod}(BG), \arrow[l, "\mathbf{oblv}_{\text{D-mod}(BG)}"'] \arrow[u, "\sigma^!"'] \end{tikzcd} \tag{6.12}$$

and according to Lemma 6.3.15, the diagram

$$\begin{tikzcd} \text{D-mod}(\mathrm{pt})_{\mathrm{rel}_{BG}} \arrow[r, "\mathbf{ind}_{\text{D-mod}(\mathrm{pt})_{\mathrm{rel}\to\mathrm{abs}}}"] & \text{D-mod}(\mathrm{pt}) \\ \mathrm{IndCoh}(BG) \arrow[u, "\sigma^!"] \arrow[r, "\mathbf{ind}_{\text{D-mod}(BG)}"] & \text{D-mod}(BG), \arrow[u, "\sigma^!"'] \end{tikzcd} \tag{6.13}$$

obtained by taking the left adjoints of the horizontal arrows, is also commutative.

6.4.2. By Sect. 6.3.6, we have

$$\text{D-mod}(\mathrm{pt})_{\mathrm{rel}_{BG}} \simeq \mathfrak{g}\text{-mod},$$

where $\mathfrak{g} := \mathrm{Lie}(G)$, and $\mathfrak{g}$-mod denotes the DG category of $\mathfrak{g}$-modules.

The functor

$$\mathbf{oblv}_{\text{D-mod}(\mathrm{pt})_{\mathrm{rel}\to\mathrm{abs}}} : \text{D-mod}(\mathrm{pt}) \to \text{D-mod}(\mathrm{pt})_{\mathrm{rel}_{BG}}$$

is the functor

$$\mathrm{triv} : \mathrm{Vect} \to \mathfrak{g}\text{-mod}$$

that sends a vector space to the $\mathfrak{g}$-module with the trivial action.

Its left adjoint

$$\mathbf{ind}_{\text{D-mod}(\mathrm{pt})_{\mathrm{rel}\to\mathrm{abs}}} : \text{D-mod}(\mathrm{pt})_{\mathrm{rel}_{BG}} \to \text{D-mod}(\mathrm{pt})$$

is the functor

$$\mathrm{conv}_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \to \mathrm{Vect}$$

of $\mathfrak{g}$-coinvariants.

6.4.3. Note that since BG is smooth, the functor

$$\Psi_{BG} : \mathrm{IndCoh}(BG) \to \mathrm{QCoh}(BG)$$

is an equivalence. We set by definition

$$\mathrm{Rep}(G) := \mathrm{QCoh}(BG).$$

6.4.4. Let us now assume that G is affine, for the duration of this subsection.

Let $BG^\bullet$ be the Čech nerve of the map $\sigma : \mathrm{pt} \to BG$. The description of $\mathrm{QCoh}(BG)$ as $\mathrm{Tot}(\mathrm{QCoh}(BG^\bullet))$ implies:

Corollary 6.4.5. *The symmetric monoidal category* $\mathrm{Rep}(G)$ *identifies with* R_G-comod, *where* R_G *is the regular representation of* G*, viewed as a cocommutative Hopf algebra.*

We claim that $\operatorname{Rep}(G)$ is in fact "the usual" DG category corresponding to the derived category of representations of G. Indeed, according to Remark 1.2.10, we have a canonical functor

$$D(\operatorname{Rep}(G)^{\heartsuit}) \to \operatorname{Rep}(G), \tag{6.14}$$

which identifies $\operatorname{Rep}(G)$ with the left-completion of $D(\operatorname{Rep}(G)^{\heartsuit})$.

However, we have:

Lemma 6.4.6. *The functor* (6.14) *is an equivalence.*

Proof. Follows from the fact that $D(\operatorname{Rep}(G)^{\heartsuit})$ is of finite cohomological dimension. □

6.4.7. Thus, we obtain that the commutative diagrams (6.12) and (6.13) identify with

$$\begin{tikzcd}
\mathfrak{g}\text{-mod} & \mathrm{Vect} \arrow[l, "\mathrm{triv}_{\mathfrak{g}}"'] \\
\operatorname{Rep}(G) \arrow[u] & \text{D-mod}(BG) \arrow[l] \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathfrak{g}\text{-mod} \arrow[r, "\mathrm{coinv}_{\mathfrak{g}}"] & \mathrm{Vect} \\
\operatorname{Rep}(G) \arrow[u] \arrow[r] & \text{D-mod}(BG), \arrow[u]
\end{tikzcd}$$

respectively. In both diagrams, the left vertical arrow is the functor

$$\operatorname{Rep}(G) \to \mathfrak{g}\text{-mod},$$

$$V \mapsto \operatorname{Res}^G_{\mathfrak{g}}\left(V \otimes \det(\mathfrak{g}^\vee)[\dim(G)]\right),$$

where $\operatorname{Res}^G_{\mathfrak{g}}$ is the usual restriction functor

$$\operatorname{Rep}(G) \to \mathfrak{g}\text{-mod}.$$

6.4.8. In particular, we obtain that the functor

$$\mathbf{ind}_{\text{D-mod}(BG)} : \operatorname{IndCoh}(BG) \to \text{D-mod}(BG),$$

composed with $\sigma^!$, identifies with the functor $\operatorname{Rep}(G) \to \mathrm{Vect}$ given by

$$V \mapsto \mathrm{coinv}_{\mathfrak{g}}\left(\operatorname{Res}^G_{\mathfrak{g}}(V \otimes \det(\mathfrak{g}^\vee)[\dim(G)])\right). \tag{6.15}$$

6.5. **Additional properties of the induction functor.** The goal of this subsection is to prove Proposition 6.5.7, which is needed for the proof of Proposition 7.1.6. As the contents of this section will not be needed elsewhere in the paper, the reader may skip it on the first pass.

6.5.1. Let $\pi : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be a schematic and quasi-compact map of algebraic stacks. For $(S, g) \in \mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}$ set

$$\widetilde{S} := \widetilde{\mathcal{Y}} \underset{\mathcal{Y}}{\times} S.$$

Let π_S and $\widetilde{g}$ denote the maps in the diagram

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\pi_S"] \arrow[d, "\widetilde{g}"'] & S \arrow[d, "g"] \\ \widetilde{\mathcal{Y}} \arrow[r, "\pi"] & \mathcal{Y}. \end{tikzcd}$$

In this case we have a naturally defined functor

$$(\pi_S)_*^{\mathrm{IndCoh}} : \mathrm{D\text{-}mod}(\widetilde{S})_{\mathrm{rel}_{\widetilde{\mathcal{Y}}}} \to \mathrm{D\text{-}mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}$$

that makes the following diagram commute:

$$\begin{tikzcd} \mathrm{D\text{-}mod}(\widetilde{S})_{\mathrm{rel}_{\widetilde{\mathcal{Y}}}} \arrow[r, "(\pi_S)_*^{\mathrm{IndCoh}}"] \arrow[d, "\mathbf{oblv}_{\mathrm{D\text{-}mod}(\widetilde{S})_{\mathrm{rel}_{\widetilde{\mathcal{Y}}}}}"'] & \mathrm{D\text{-}mod}(S)_{\mathrm{rel}_{\mathcal{Y}}} \arrow[d, "\mathbf{oblv}_{\mathrm{D\text{-}mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}}"] \\ \mathrm{IndCoh}(\widetilde{S}) \arrow[r, "\pi_S^{\mathrm{IndCoh},*}"] & \mathrm{IndCoh}(S). \end{tikzcd}$$

The following isomorphism generalizes the base-change isomorphism for IndCoh:

Lemma 6.5.2. *There exists a canonical isomorphism*

$$g^! \circ \pi_*^{\mathrm{IndCoh}} \simeq (\pi_S)_*^{\mathrm{IndCoh}} \circ \widetilde{g}^!$$

as functors $\mathrm{IndCoh}(\widetilde{\mathcal{Y}}) \to \mathrm{D\text{-}mod}(S)_{\mathrm{rel}_{\mathcal{Y}}}$.

6.5.3. In addition, we have the following assertion that generalizes the commutative diagram (5.11):

Lemma 6.5.4. *The diagram*

$$\begin{tikzcd} \mathrm{D\text{-}mod}(\widetilde{S}) \arrow[r, "(\pi_S)_{\mathrm{dR},*}"] & \mathrm{D\text{-}mod}(S) \\ \mathrm{D\text{-}mod}(\widetilde{S})_{\mathrm{rel}_{\widetilde{\mathcal{Y}}}} \arrow[r, "(\pi_S)_*^{\mathrm{IndCoh}}"] \arrow[u, "\mathbf{ind}_{\mathrm{D\text{-}mod}(\widetilde{S})_{\mathrm{rel}\to\mathrm{abs}}}"] & \mathrm{D\text{-}mod}(S)_{\mathrm{rel}_{\mathcal{Y}}} \arrow[u, "\mathbf{ind}_{\mathrm{D\text{-}mod}(S)_{\mathrm{rel}\to\mathrm{abs}}}"'] \end{tikzcd}$$

canonically commutes.

Remark 6.5.5. In terms of the formalism that will be explained in [GR2], the commutativity of the diagram in Lemma 6.5.4 follows by taking direct images on IndCoh along the morphisms in the following diagram

$$\begin{tikzcd} \widetilde{S}_{\mathrm{dR}} \underset{\widetilde{\mathcal{Y}}_{\mathrm{dR}}}{\times} \widetilde{\mathcal{Y}} \arrow[r] \arrow[d] & S_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y} \arrow[d] \\ \widetilde{S}_{\mathrm{dR}} \arrow[r] & S_{\mathrm{dR}}. \end{tikzcd}$$

6.5.6. We now claim:

Proposition 6.5.7. *For the map $\pi : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ as above, the following diagram of functors canonically commutes:*

$$\begin{tikzcd}
\text{D-mod}(\widetilde{\mathcal{Y}}) \arrow[r, "\pi_{\mathrm{dR},*}"] & \text{D-mod}(\mathcal{Y}) \\
\mathrm{IndCoh}(\widetilde{\mathcal{Y}}) \arrow[u, "\mathbf{ind}_{\text{D-mod}(\widetilde{\mathcal{Y}})}"] \arrow[r, "\pi_*^{\mathrm{IndCoh}}"] & \mathrm{IndCoh}(\mathcal{Y}). \arrow[u, "\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}"']
\end{tikzcd}$$

Proof. By Lemma 6.3.5, we need to show that for every $(S,g) \in \mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}}$ the functors

$$(6.16) \qquad g^! \circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ \pi_*^{\mathrm{IndCoh}} \text{ and } g^! \circ \pi_{\mathrm{dR},*} \circ \mathbf{ind}_{\text{D-mod}(\widetilde{\mathcal{Y}})}, \quad \mathrm{IndCoh}(\widetilde{\mathcal{Y}}) \to \text{D-mod}(S)$$

are canonically isomorphic.

We rewrite the left-hand side in (6.16) as

$$\mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}} \circ g^! \circ \pi_*^{\mathrm{IndCoh}} \overset{\text{Lemma 6.5.2}}{\simeq}$$
$$\simeq \mathbf{ind}_{\text{D-mod}(S)_{\mathrm{rel}\to\mathrm{abs}}} \circ (\pi_S)_*^{\mathrm{IndCoh}} \circ \widetilde{g}^! \overset{\text{Lemma 6.5.4}}{\simeq}$$
$$\simeq (\pi_S)_{\mathrm{dR},*} \circ \mathbf{ind}_{\text{D-mod}(\widetilde{S})_{\mathrm{rel}\to\mathrm{abs}}} \circ \widetilde{g}^! \simeq$$
$$\simeq (\pi_S)_{\mathrm{dR},*} \circ \widetilde{g}^! \circ \mathbf{ind}_{\text{D-mod}(\widetilde{\mathcal{Y}})} \simeq g^! \circ \pi_{\mathrm{dR},*} \circ \mathbf{ind}_{\text{D-mod}(\widetilde{\mathcal{Y}})},$$

as required. □

Remark 6.5.8. In terms of the formalism of [GR2], the assertion of Proposition 6.5.7 follows by taking direct images along the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{Y}} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\widetilde{\mathcal{Y}}_{\mathrm{dR}} \arrow[r] & \mathcal{Y}_{\mathrm{dR}}.
\end{tikzcd}$$

7. De Rham cohomology on an algebraic stack

7.1. **Definition of De Rham cohomology.**

7.1.1. The presentation of $\text{D-mod}(\mathcal{Y})$ as in (6.5) and Sect. 5.2.9 imply that there exists a canonically defined object

$$k_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y}),$$

such that for every smooth morphism $g : S \to \mathcal{Y}$ with S being a DG scheme one has

$$g^*_{\mathrm{dR}}(k_{\mathcal{Y}}) = k_Z.$$

We define the *not necessarily continuous* functor $\Gamma_{\mathrm{dR}}(\mathcal{Y}, -) : \text{D-mod}(\mathcal{Y}) \to \mathrm{Vect}$ as

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M}) := \mathcal{M}aps_{\text{D-mod}(\mathcal{Y})}(k_{\mathcal{Y}}, -).$$

7.1.2. By (6.5), the functor $\Gamma_{\mathrm{dR}}(\mathcal{Y},-)$ can be calculated as follows: for $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$, we have:

$$\Gamma_{\mathrm{dR}}(\mathcal{Y},\mathcal{M}) \simeq \underset{(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}}{\underleftarrow{lim}} \Gamma_{\mathrm{dR}}(S, g^*_{\mathrm{dR}}(\mathcal{M})). \tag{7.1}$$

More economically, for a given smooth atlas $f : Z \to \mathcal{Y}$, by (6.6) we have:

$$\Gamma_{\mathrm{dR}}(\mathcal{Y},\mathcal{M}) \simeq \mathrm{Tot}\left(\Gamma_{\mathrm{dR}}(Z^\bullet/\mathcal{Y}, \mathcal{M}|_{Z^\bullet/\mathcal{Y}})\right),$$

where $\mathcal{M}|_{Z^\bullet/\mathcal{Y}}$ again denotes the $(\mathrm{dR},*)$-pullback.

7.1.3. *Warning.* Even if $\mathcal{Y}$ is quasi-compact and moreover, even if $\mathcal{Y}$ is QCA, the functor $\Gamma_{\mathrm{dR}}(\mathcal{Y},-)$ is *not necessarily continuous* (see the Examples in Sect. 7.1.4 and Sect. 7.2 below), which means that the object $k_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y})$ is *not necessarily compact.*

See also Corollary 10.2.6 and Definition 10.2.2 below for a characterization of those quasi-compact stacks $\mathcal{Y}$ for which the functor $\Gamma_{\mathrm{dR}}(\mathcal{Y},-)$ is continuous.

7.1.4. *Example.* Let $\mathcal{Y} := B\mathbb{G}_m$. Let us show that the functor $\Gamma_{\mathrm{dR}}(\mathcal{Y},-)$ is not continuous. Let A denote the graded algebra formed by $\mathrm{Ext}^i(k_{\mathcal{Y}}, k_{\mathcal{Y}}) = H^i(\Gamma_{\mathrm{dR}}(\mathcal{Y}, k_{\mathcal{Y}}))$.

It is easy to see that $A = k[u]$, where $\deg u = 2$. The diagram

$$k_{\mathcal{Y}} \xrightarrow{u} k_{\mathcal{Y}}[2] \xrightarrow{u} k_{\mathcal{Y}}[4] \xrightarrow{u} \ldots \tag{7.2}$$

has a zero colimit: the pullback functor under $\mathrm{pt} \to B\mathbb{G}_m$ is conservative and continuous, and the pullback of (7.2) to pt consists of zero maps.

However, when we apply the functor $\Gamma_{\mathrm{dR}}(\mathcal{Y},-)$ to (7.2) we obtain the diagram

$$A \xrightarrow{u} A[2] \xrightarrow{u} A[4] \xrightarrow{u} \ldots$$

whose colimit is nonzero.

7.1.5. The following key calculation will be performed in Sect. 7.9:

Proposition 7.1.6. *For* $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y})$ *there exists a canonical isomorphism*

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})) \simeq \Gamma^{\mathrm{IndCoh}}(\mathcal{Y},\mathcal{F}).$$

7.2. **Example: classifying stacks.** In this subsection we shall analyze the example of $\mathcal{Y} = BG$, where G is a connected algebraic group. In particuar, we will show that if G in non-unipotent, then $\Gamma_{\mathrm{dR}}(BG,-)$ is not continuous.

7.2.1. Assume first that G is affine. Then "non-unipotent" means that G contains a copy of $\mathbb{G}_m$. The morphism

$$\pi : B\mathbb{G}_m \to BG$$

is schematic and quasi-compact, so the functor $\pi_{\mathrm{dR},*}$ is continuous. We have

$$\Gamma_{\mathrm{dR}}(B\mathbb{G}_m,-) \simeq \Gamma_{\mathrm{dR}}(BG,-) \circ \pi_{\mathrm{dR},*}.$$

In particular, the Example in Sect. 7.1.4 implies that the functor $\Gamma_{\mathrm{dR}}(BG,-)$ is non-continuous.

7.2.2. For a general connected G, let us describe the category D-mod(BG) explicitly. Recall (see Sect. 6.4) that σ denotes the morphism

$$\mathrm{pt} \to BG.$$

The functor $\sigma^!$ admits a left adjoint, denoted $\sigma_!$.

Since $\sigma^!$ is conservative, and both functors are continuous, the Barr-Beck-Lurie theorem (see e.g. [GL:DG, Sect. 3.1.2]), implies that the category D-mod(BG) identifies with that of modules for the monad $\sigma^! \circ \sigma_!$ acting on D-mod(pt) = Vect.

The above monad identifies with the associative algebra in Vect

$$B := \left(\mathcal{M}aps_{\text{D-mod}(BG)}(\sigma_!(k), \sigma_!(k))\right)^{\mathrm{op}}.$$

Hence, we obtain an equivalence of categories

$$\text{D-mod}(BG) \simeq B\text{-mod}, \tag{7.3}$$

where $B \in B$-mod corresponds to the object $\sigma_!(k) \in$ D-mod(BG), which is a compact generator of this category.

7.2.3. By Verdier duality

$$B \simeq \left(\Gamma_{\mathrm{dR}}(G, k_G)\right)^\vee, \tag{7.4}$$

where the algebra structure on the right-hand side is given by the product operation $G \times G \to G$. It is well-known that, unless G is unipotent, B is isomorphic to the exterior algebra on generators in degrees $-(2m_i - 1)$, $m_i \in \mathbb{Z}^{>0}$, where i runs through some finite set.

The presentation of B given by (7.4) shows that the structure of associative algebra on B canonically upgrades to that of co-commutative Hopf algebra. In particular, B is augmented.

The augmentation module

$$k \in B\text{-mod}$$

corresponds to the object $k_{BG} \in$ D-mod(BG). In terms of (7.4), the augmentation corresponds to the map $p_G : G \to \mathrm{pt}$.

7.2.4. We obtain that the algebra

$$A := \mathcal{M}aps_{\text{D-mod}(BG)}(k_{BG}, k_{BG})$$

is canonically isomorphic to the *Koszul dual* of B, i.e.,

$$A \simeq \mathcal{M}aps_{B\text{-mod}}(k, k).$$

Explicitly, A is a polynomial algebra on generators in degrees $2m_1, ..., 2m_r$ for m_i as above. [21]

In particular, this shows that k is not a compact object in B-mod (otherwise A would have been finite-dimensional).

The functor $\Gamma_{\mathrm{dR}}(BG, -)$ is given, in terms of (7.3), by

$$M \mapsto \mathcal{M}aps_B(k, M),$$

so it is not continuous.

[21] Over $\mathbb{C}$, the latter observation reproduces a well-known fact about the cohomology of the classifying space.

7.2.5. Let us assume once again that G is affine, and compare the above description of the category D-mod(BG) with Sect. 6.4.

We obtain a pair of commutative diagrams

(7.5)
$$\begin{tikzcd} \mathfrak{g}\text{-mod} & \mathrm{Vect} \arrow[l, "\mathrm{triv}_{\mathfrak{g}}"'] \\ \mathrm{Rep}(G) \arrow[u] & B\text{-mod}, \arrow[l, "\mathbf{oblv}_{\text{D-mod}(BG)}"'] \arrow[u] \end{tikzcd}$$

and

(7.6)
$$\begin{tikzcd} \mathfrak{g}\text{-mod} \arrow[r, "\mathrm{coinv}_{\mathfrak{g}}"] & \mathrm{Vect} \\ \mathrm{Rep}(G) \arrow[u] \arrow[r, "\mathbf{ind}_{\text{D-mod}(BG)}"] & B\text{-mod}. \arrow[u] \end{tikzcd}$$

Note that there is a natural forgetful functor

$$B\text{-mod} \to \mathrm{Rep}(G) \tag{7.7}$$

corresponding to the homomorphism of Hopf algebras

$$B^\vee = \Gamma_{\mathrm{dR}}(G, k_G) \simeq \mathcal{M}aps_{\text{D-mod}(G)}(\omega_G, \omega_G) \to \mathcal{M}aps_{\mathrm{IndCoh}(G)}(\omega_G, \omega_G) \simeq R_G. \tag{7.8}$$

It is easy to see that the functor

$$\mathbf{oblv}_{\text{D-mod}(BG)} : B\text{-mod} \to \mathrm{Rep}(G)$$

in (7.5) equals the composition of the functor (7.7), followed by the functor

$$V \mapsto V \otimes \det(\mathfrak{g})[-\dim(G)] : \mathrm{Rep}(G) \to \mathrm{Rep}(G).$$

In particular, we obtain that the functor

$$V \mapsto \mathrm{coinv}_{\mathfrak{g}}(\mathrm{Res}^G_{\mathfrak{g}}(V)) : \mathrm{Rep}(G) \to \mathrm{Vect}$$

canonically factors as

$$\mathrm{Rep}(G) \to B\text{-mod} \overset{\mathbf{oblv}_B}{\longrightarrow} \mathrm{Vect},$$

where the functor $\mathrm{Rep}(G) \to B$-mod is the left adjoint to the functor in (7.7).

Remark 7.2.6. Let $BG^\bullet_{\mathrm{dR}}$ denote the simplicial object of PreStk obtained by applying the functor $\mathcal{Y} \mapsto \mathcal{Y}_{\mathrm{dR}}$ to $BG^\bullet$. Equivalently, $BG^\bullet_{\mathrm{dR}}$ is the Čech nerve of the map $\mathrm{pt} \to (BG)_{\mathrm{dR}}$.

We have:

$$\mathrm{Tot}(\mathrm{QCoh}(BG^\bullet_{\mathrm{dR}})) \simeq \mathrm{Tot}(\mathrm{IndCoh}(BG^\bullet_{\mathrm{dR}})) \simeq \mathrm{Tot}(\text{D-mod}(BG^\bullet)) \simeq \text{D-mod}(BG),$$

where the latter isomorphism is given by (6.6).

Set by definition

$$\mathrm{Rep}(G_{\mathrm{dR}}) := \mathrm{Tot}(\mathrm{QCoh}(BG^\bullet_{\mathrm{dR}})).$$

We can informally interpret the resulting adjunction $\mathrm{Rep}(G) \rightleftarrows \mathrm{Rep}(G_{\mathrm{dR}})$ as coming from the short exact sequence

$$1 \to \mathfrak{g} \to G \to G_{\mathrm{dR}} \to 1.$$

The latter will be made precise in [GR2] by considering the formal completion $G^\wedge$ at $1 \in G$, and showing that $G_{\mathrm{dR}} \simeq G/G^\wedge$ and that proving that

$$\mathfrak{g}\text{-mod} \simeq \mathrm{Rep}(G^\wedge) := \mathrm{Tot}(\mathrm{QCoh}((G^\wedge)^\bullet)).$$

7.3. **Coherence and compactness on algebraic stacks.**

7.3.1. Let

$$\text{D-mod}_{\mathrm{coh}}(\mathcal{Y}) \subset \text{D-mod}(\mathcal{Y})$$

be the full subcategory consisting of objects $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$ such that $g^!(\mathcal{M}) \in \text{D-mod}_{\mathrm{coh}}(S)$ for any smooth map $g : S \to \mathcal{Y}$, where S is a DG scheme. (Of course, $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ is not cocomplete.)

It is easy to see that coherence condition is equivalent to requiring that $g^*_{\mathrm{dR}}(\mathcal{M}) \in \text{D-mod}(S)$ belong to $\text{D-mod}_{\mathrm{coh}}(S)$ for any smooth map $g : S \to \mathcal{Y}$, where S is a DG scheme. (Indeed, for smooth maps, g^*_{dR} and $g^!$ differ by a cohomological shift on each connected component of S.)

It is also clear, that in either definition it suffices to consider those S that are quasi-compact, or even affine.

Finally, it is enough to require either of the above conditions for just one smooth atlas $f : Z \to \mathcal{Y}$.

7.3.2. The object $k_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y})$ is always in $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$. On the other hand, even if $\mathcal{Y}$ is quasi-compact it may happen that $k_{\mathcal{Y}}$ is not compact (see Sect.7.1.3).

So it is not true that $\text{D-mod}(\mathcal{Y})^c$ *equals* $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ *for any quasi-compact stack.* [22] However, we have:

Lemma 7.3.3. *For any algebraic stack* $\mathcal{Y}$ *one has the inclusion*

$$\text{D-mod}(\mathcal{Y})^c \subset \text{D-mod}_{\mathrm{coh}}(\mathcal{Y})\,. \tag{7.9}$$

Proof. The proof repeats verbatim that of Proposition 3.4.2(a):

We need to show that if $\mathcal{M} \in \text{D-mod}(\mathcal{Y})^c$ then for any smooth map $g : S \to \mathcal{Y}$ with S being a quasi-compact (or even affine) DG scheme, one has $g^*_{\mathrm{dR}}(\mathcal{M}) \in \text{D-mod}_{\mathrm{coh}}(S) = \text{D-mod}(S)^c$.

However, this is clear since g^*_{dR} admits a right adjoint that commutes with colimits, namely $g_{\mathrm{dR},*}$ (see Sect. 6.1.7). □

7.3.4. *Verdier duality on algebraic stacks.* Let us observe that there exists a canonical involutive anti self-equivalence

$$\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}} : (\text{D-mod}_{\mathrm{coh}}(\mathcal{Y}))^{\mathrm{op}} \to \text{D-mod}_{\mathrm{coh}}(\mathcal{Y}) \tag{7.10}$$

(called Verdier duality) such that for any smooth map $g : S \to \mathcal{Y}$ from a scheme, we have:

$$g^! \circ \mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}} \simeq \mathbb{D}^{\mathrm{Verdier}}_{S} \circ g^*_{\mathrm{dR}}.$$

In other words, to define (7.10) we use two different realizations of $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ as a limit: the one of (6.1) for the first copy of $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$, and the one of (6.5) for the second one.

Lemma 7.3.5. *For any* $\mathcal{M} \in \text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ *and* $\mathcal{M}' \in \text{D-mod}(\mathcal{Y})$ *one has a canonical isomorphism*

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{Y})}(\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}}(\mathcal{M}), \mathcal{M}') \simeq \Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}').$$

[22] According to Corollary 10.2.7 below, $\text{D-mod}(\mathcal{Y})^c = \text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ for those quasi-compact stacks that are *safe* in the sense of Definition 10.2.2.

Proof. The two sides are calculated as limits over $(S,g)\in(\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}$ of

$$\mathcal{M}aps_{\text{D-mod}(S)}\left(g^!(\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}}(\mathcal{M})),g^!(\mathcal{M}')\right) \text{ and } \Gamma_{\mathrm{dR}}\left(S,g^*_{\mathrm{dR}}(\mathcal{M}\overset{!}{\otimes}\mathcal{M}')\right),$$

respectively. By (5.14), we have

$$\mathcal{M}aps_{\text{D-mod}(S)}\left(g^!(\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}}(\mathcal{M})),g^!(\mathcal{M}')\right)\simeq\mathcal{M}aps_{\text{D-mod}(S)}\left(\mathbb{D}^{\mathrm{Verdier}}_{S}(g^*_{\mathrm{dR}}(\mathcal{M})),g^!(\mathcal{M}')\right)\simeq$$
$$\simeq\Gamma_{\mathrm{dR}}\left(S,g^*_{\mathrm{dR}}(\mathcal{M})\overset{!}{\otimes}g^!(\mathcal{M}')\right),$$

so the required isomorphism follows from (6.3). $\square$

Combining Lemma 7.3.5, Proposition 4.4.4, Lemma 6.3.20 and Proposition 7.1.6, we obtain:

Corollary 7.3.6. *If $\mathcal{F}\in\mathrm{Coh}(\mathcal{Y})$ then $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})\in\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$, and we have:*

$$\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}}\left(\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})\right)\simeq\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}\left(\mathbb{D}^{\mathrm{Serre}}_{\mathcal{Y}}(\mathcal{F})\right). \tag{7.11}$$

7.4. $(\mathrm{dR},*)$**-pushforwards for stacks.**

7.4.1. If $\pi:\mathcal{Y}_1\to\mathcal{Y}_2$ is a map between algebraic stacks. We define the functor

$$\pi_{\mathrm{dR},*}:\text{D-mod}(\mathcal{Y}_1)\to\text{D-mod}(\mathcal{Y}_2)$$

by

$$\pi_{\mathrm{dR},*}(\mathcal{M}):=\underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\underset{\longleftarrow}{lim}}(\pi\circ g)_{\mathrm{dR},*}(g^*_{\mathrm{dR}}(\mathcal{M})), \tag{7.12}$$

where $(\pi\circ g)_{\mathrm{dR},*}$ is understood in the sense of Sect. 6.1.7.

Remark 7.4.2. Unfortunately, we do not know how to characterize the functor $\pi_{\mathrm{dR},*}$ intrinsically. Unless π is smooth (or, more generally, locally acyclic in an appropriate sense), the left adjoint to $\pi_{\mathrm{dR},*}$ will not be defined as a functor $\text{D-mod}(\mathcal{Y}_2)\to\text{D-mod}(\mathcal{Y}_1)$, but rather on the corresponding pro-categories.

See, however, Corollary 7.8.4, which gives an explicit formula for maps into $\pi_{\mathrm{dR},*}(-)$ out of a coherent object of $\text{D-mod}(\mathcal{Y}_2)$.

7.4.3. *Warning.* The functor $\pi_{\mathrm{dR},*}$ has features similar to those of the functor π_* discussed in Sect. 1.3.1. For a general morphism π, *it is not continuous* (see Sect. 7.1.3); it does not satisfy base change (even for open embeddings) or the projection formula (see Sects. 7.6 and 7.7 below for the explanation of what this means).

That said, the restriction of $\pi_{\mathrm{dR},*}$ to $\text{D-mod}(\mathcal{Y}_1)^+$ behaves reasonably, as is guaranteed by Proposition 7.6.8.

However, on all of $\text{D-mod}(\mathcal{Y}_1)$, the functor $\pi_{\mathrm{dR},*}$ may surprise one's intuition; see Sect. 7.8.7 for a particularly treacherous example.

7.5. **Properties of the** $(\mathrm{dR},*)$**-pushforward.** This subsection is devoted to proving that $\pi_{\mathrm{dR},*}$ has *some* reasonable properties. As the following discussion is purely technical (and will amount to showing that certain limits can be commuted with certain colimits), the reader can skip it on the first pass, and return to it when necessary.

7.5.1. One can calculate $\pi_{\mathrm{dR},*}$ more economically as follows.

Let A be a category equipped with a functor
$$a \mapsto (S_a, g_a) : A \to (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}}$$
with the property that the functor given by $(\mathrm{dR},*)$-pullback
$$\text{D-mod}(\mathcal{Y}_1) \to \underset{a\in A^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} \text{D-mod}(S_a),$$
is an equivalence, cf. Sect. 1.2.5.

In Sect. 7.8.8 we will prove:

Lemma 7.5.2. *For $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$, the map*
$$\pi_{\mathrm{dR},*}(\mathcal{M}_1) \to \underset{a\in A^{\mathrm{op}}}{\underset{\longleftarrow}{lim}} (\pi\circ g_\alpha)_{\mathrm{dR},*}\circ (g_\alpha)^*_{\mathrm{dR}}(\mathcal{M}_1)$$
is an isomorphism.

7.5.3. Assume for a moment that π is schematic and quasi-compact. In this case we obtain two functors, both denoted $\pi_{\mathrm{dR},*}$. One such functor, which we shall temporarily denote by $\pi^{(a)}_{\mathrm{dR},*}$ was introduced in Sect. 6.1.7 and was specific to schematic quasi-compact maps. Another functor, which we shall temporarily denote by $\pi^{(b)}_{\mathrm{dR},*}$ is the one from (7.12).

It is easy to see that there is a natural transformation
$$\pi^{(a)}_{\mathrm{dR},*} \to \pi^{(b)}_{\mathrm{dR},*}. \tag{7.13}$$

We claim:

Proposition 7.5.4. *The natural transformation* (7.13) *is an isomorphism.*

Due to this proposition, we obtain that the notation $\pi_{\mathrm{dR},*}$ is unambiguous.

Proof. For $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$ we calculate $\pi^{(b)}_{\mathrm{dR},*}(\mathcal{M}_1)$ by Lemma 7.5.2 via the category $A = (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_2,\mathrm{smooth}}$, see Lemma 1.2.6.

For
$$(S_2, g_2) \in (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_2,\mathrm{smooth}}$$
consider the Cartesian diagram
$$\begin{tikzcd} S_1 \arrow[r, "g_1"] \arrow[d, "\pi_S"'] & \mathcal{Y}_1 \arrow[d, "\pi"] \\ S_2 \arrow[r, "g_2"] & \mathcal{Y}_2, \end{tikzcd}$$
and we have:
$$(\pi\circ g_1)_{\mathrm{dR},*}\circ (g_1)^*_{\mathrm{dR}}(\mathcal{M}_1) \simeq (g_2\circ \pi_S)_{\mathrm{dR},*}\circ (g_1)^*_{\mathrm{dR}}(\mathcal{M}_1) \simeq (g_2)_{\mathrm{dR},*}\circ (\pi_S)_{\mathrm{dR},*}\circ (g_1)^*_{\mathrm{dR}}(\mathcal{M}_1).$$

However, it is easy to see that the natural transformation
$$(g_2)^*_{\mathrm{dR}}\circ \pi^{(a)}_{\mathrm{dR},*} \to (\pi_S)_{\mathrm{dR},*}\circ (g_1)^*_{\mathrm{dR}}$$
arising by adjunction is an isomorphism.

Hence,
$$(\pi\circ g_1)_{\mathrm{dR},*}\circ (g_1)^*_{\mathrm{dR}}(\mathcal{M}_1) \simeq (g_2)_{\mathrm{dR},*}\circ (g_2)^*_{\mathrm{dR}}\circ \pi^{(a)}_{\mathrm{dR},*}(\mathcal{M}_1).$$

Passing to the limit over (S_2, g_2) we obtain the desired isomorphism. $\square$

7.5.5. *Transitivity.* We note the following property of the functor $\pi_{\mathrm{dR},*}$:

Lemma 7.5.6. *There exists a canonical isomorphism of (non-contunuous) functors*

$$\text{D-mod}(\mathcal{Y}_1) \to \mathrm{Vect} : \Gamma_{\mathrm{dR}}(\mathcal{Y}_1, -) \simeq \Gamma_{\mathrm{dR}}(\mathcal{Y}_2, \pi_{\mathrm{dR},*}(-)).$$

Proof. Follows from the fact that the partially defined left adjoint π^*_{dR} of $\pi_{\mathrm{dR},*}$ is defined on $k_{\mathcal{Y}_2}$ and

$$\pi^*_{\mathrm{dR}}(k_{\mathcal{Y}_2}) \simeq k_{\mathcal{Y}_1}.$$

□

Let now $\phi : \mathcal{Y}_2 \to \mathcal{Y}_3$ be another morphism between algebraic stacks. It is easy to see that there exists a natural transformation

$$\phi_{\mathrm{dR},*} \circ \pi_{\mathrm{dR},*} \to (\phi \circ \pi)_{\mathrm{dR},*}. \tag{7.14}$$

The natural transformation is not always an isomorphism, see Sect. 7.8.7 for a counterexample. In what follows we shall need the following statement, proved in Sect. 7.8.5:

Proposition 7.5.7. *Suppose that π is schematic and quasi-compact. Then the natural transformation* (7.14) *is an isomorphism.*

We refer the reader to Sect. 7.8.6 where several more situations are given, in which (7.14) is an isomorphism.

7.5.8. *Pushforward of induced D-modules.* Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be as above. It is easy to see that there exists a canonical natural transformation between functors $\mathrm{IndCoh}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2)$, namely,

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Y}_2)} \circ \pi^{\mathrm{IndCoh}}_{\text{non-ren},*} \to \pi_{\mathrm{dR},*} \circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_1)}. \tag{7.15}$$

The following assertion will be proved in Sect. 9.3.17:

Proposition 7.5.9. *Suppose that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are QCA. Then the natural transformation* (7.15) *is an isomorphism.*

Remark 7.5.10. We do not know whether (7.15) is an isomorphism for an arbitrary morphism of stacks. Note, however, that when $\mathcal{Y}_2 = \mathrm{pt}$, this is true by Proposition 7.1.6.

7.5.11. Note that if π is smooth, the functor $\pi_{\mathrm{dR},*}$ commutes with limits, since it admits a left adjoint. In general, this will not be so. However, we have the following useful property:

Lemma 7.5.12. *For $\mathcal{M} \in \text{D-mod}(\mathcal{Y}_1)$, the natural map*

$$\pi_{\mathrm{dR},*}(\mathcal{M}) \to \underset{\underset{n}{\longleftarrow}}{lim}\, \pi_{\mathrm{dR},*}(\tau^{\geq -n}(\mathcal{M}))$$

is an isomorphism.

Proof. We have:

$$\underset{\underset{n}{\longleftarrow}}{lim}\, \pi_{\mathrm{dR},*}(\tau^{\geq -n}(\mathcal{M})) \simeq \underset{\underset{n}{\longleftarrow}}{lim} \underset{\underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\longleftarrow}}{lim} (\pi \circ g)_{\mathrm{dR},*}(g^*_{\mathrm{dR}}(\tau^{\geq -n}(\mathcal{M}))) \simeq$$

$$\simeq \underset{\underset{(S,g)\in((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}}{\longleftarrow}}{lim} \underset{\underset{n}{\longleftarrow}}{lim}\, (\pi \circ g)_{\mathrm{dR},*}(g^*_{\mathrm{dR}}(\tau^{\geq -n}(\mathcal{M}))).$$

We claim that for each (S, g), the map

$$(\pi \circ g)_{\mathrm{dR},*}(g^*_{\mathrm{dR}}(\mathcal{M})) \to \underset{\underset{n}{\longleftarrow}}{lim}\, (\pi \circ g)_{\mathrm{dR},*}(g^*_{\mathrm{dR}}(\tau^{\geq -n}(\mathcal{M})))$$

is an isomorphism.

First, since g^*_{dR} is of bounded cohomological amplitude, we rewrite

$$\underset{n}{\underleftarrow{lim}}\,(\pi\circ g)_{\mathrm{dR},*}(g^*_{\mathrm{dR}}(\tau^{\geq -n}(\mathcal{M})))\simeq \underset{n}{\underleftarrow{lim}}\,(\pi\circ g)_{\mathrm{dR},*}(\tau^{\geq -n}(g^*_{\mathrm{dR}}(\mathcal{M}))).$$

Thus, we have reduced the assertion of the lemma to the case when $\mathcal{Y}_1=S$ is a quasi-compact DG scheme. In this case, the functor $\pi_{\mathrm{dR},*}$ has itself a bounded cohomological amplitude, so

$$\underset{n}{\underleftarrow{lim}}\,\pi_{\mathrm{dR},*}(\tau^{\geq -n}(\mathcal{M}))\simeq \underset{n}{\underleftarrow{lim}}\,\tau^{\geq -n}(\pi_{\mathrm{dR},*}(\mathcal{M})).$$

Now, the desired assertion follows from the left-completeness of D-mod$(\mathcal{Y}_2)$ in its t-structure. $\square$

7.6. **Base change for the** $(\mathrm{dR},*)$**-pushforward.**

7.6.1. Let $\pi:\mathcal{Y}_1\to\mathcal{Y}_2$ be as above, and let $\phi_2:\mathcal{Y}_2'\to\mathcal{Y}_2$ be another morphism of algebraic stacks.

Consider the Cartesian diagram

$$\begin{tikzcd} \mathcal{Y}_1' \arrow[r, "\phi_1"] \arrow[d, "\pi'"'] & \mathcal{Y}_1 \arrow[d, "\pi"] \\ \mathcal{Y}_2' \arrow[r, "\phi_2"] & \mathcal{Y}_2. \end{tikzcd}$$

For $\mathcal{M}_1\in$ D-mod$(\mathcal{Y}_1)$ there exists a canonically defined map

$$\phi_2^!\circ\pi_{\mathrm{dR},*}(\mathcal{M}_1)\to\pi'_{\mathrm{dR},*}\circ\phi_1^!(\mathcal{M}_1). \tag{7.16}$$

Definition 7.6.2.

(a) *The triple* $(\phi_2,\mathcal{M}_1,\pi)$ *satisfies base change if the map* (7.16) *is an isomorphism.*

(b) *The pair* $(\mathcal{M}_1,\pi)$ *satisfies base change if* (7.16) *is an isomorphism for any* ϕ_2*.*

(c) *The morphism* π *satisfies base change if* (7.16) *is an isomorphism for any* ϕ_2 *and* $\mathcal{M}_1$*.*

7.6.3. We have the following analog of Proposition 1.3.6:

Proposition 7.6.4. *Given* $\pi:\mathcal{Y}_1\to\mathcal{Y}_2$*, for* $\mathcal{M}_1\in$ D-mod$(\mathcal{Y}_1)$ *the following conditions are equivalent:*

(i) $(\mathcal{M}_1,\pi)$ *satisfies base change.*

(ii) $(\phi_2,\mathcal{M}_1,\pi)$ *satisfies base change whenever* $\mathcal{Y}_2=S_2\in\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$.

(iii) *For any* $S_2'\overset{f_2}{\to}S_2\overset{g_2}{\to}\mathcal{Y}_2$ *with* $S_2,S_2'\in\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$*, the triple* $(f_2,\mathcal{M}_{S,1},\pi_S)$ *satisfies base change, where*

$$g_1:S_2\underset{\mathcal{Y}_2}{\times}\mathcal{Y}_1\to\mathcal{Y}_1,\ \mathcal{M}_{S,1}:=g_1^!(\mathcal{M}_1)\ \text{and}\ \pi_S:S_2\underset{\mathcal{Y}_2}{\times}\mathcal{Y}_1\to S_2.$$

Proof. Follows in the same way as Proposition 1.3.6 using the following observation.

Let $i\mapsto\mathbf{C}^i, i\in I$ be a family of cocomplete DG categories, let $F^{i,j}:\mathbf{C}^i\to\mathbf{C}^j$ denote the corresponding family of functors. Let $\mathbf{C}:=\underset{i}{\underleftarrow{lim}}\,\mathbf{C}^i$ be their limit.

For another category of indices A, let $\mathbf{c}_a, a \in A$ be an A-family of objects in $\mathbf{C}$, i.e., a compatible family of objects $\mathbf{c}^i_a \in \mathbf{C}^i, a \in A$. Denote

$$\mathbf{c}^i := \underset{\overleftarrow{a}}{lim}\, \mathbf{c}^i_a \in \mathbf{C}^i$$

(recall that cocomplete DG categories are closed under limits, see Sect. 0.6.3).

Lemma 7.6.5. *Suppose that the maps $F_{i,j}(\mathbf{c}^i) \to \mathbf{c}^j$ are isomorphisms. Then*

$$\mathbf{c} := \underset{\overleftarrow{a}}{lim}\, \mathbf{c}_a \in \mathbf{C}$$

corresponds to the system $i \mapsto \mathbf{c}^i$.

We apply this lemma as follows: the category of indices I is $((\mathrm{DGSch})^{\mathrm{aff}}_{/\mathcal{Y}_2})^{\mathrm{op}}$ and for an object $i = (Z, f) \in I$, set

$$\mathbf{C}^i = \text{D-mod}(Z),$$

so that $\mathbf{C} = \text{D-mod}(\mathcal{Y}_2)$.

We take the category of indices A to be $((\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathcal{Y}_1,\mathrm{smooth}})^{\mathrm{op}}$. For each $a = (S, g) \in A$ we set

$$\mathbf{c}_a := (\pi \circ g)_{\mathrm{dR},*}(g^*_{\mathrm{dR}}(\mathcal{M}_1)),$$

so that $\mathbf{c} = \pi_{\mathrm{dR},*}(\mathcal{M})$ and for $i = (Z, f)$

$$\mathbf{c}^i = (\pi_Z)_{\mathrm{dR},*}(\widetilde{f}^!(\mathcal{M}_1)),$$

where

$$\begin{tikzcd} Z \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_Z"'] & \mathcal{Y}_1 \arrow[d, "\pi"] \\ Z \arrow[r, "f"] & \mathcal{Y}_2. \end{tikzcd}$$

□

It is clear that schematic quasi-compact morphisms satisfy base change.

Remark 7.6.6. In Theorem 10.2.4 we shall show that any morphism which is *safe* satisfies base change. Furthermore, in Corollary 9.3.14 we will show that for a morphism between QCA stacks, a pair $(\mathcal{M}_1, \pi)$ satisfies base change if $\mathcal{M}_1$ is *safe*.

7.6.7. As in Corollary 1.3.17 we have:

Proposition 7.6.8. *Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a quasi-compact morphism between algebraic stacks. Then:*

(a) *For any $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)^+$, the pair $(\mathcal{M}_1, \pi)$ satisfies base change.* [23]

(b) *If $\mathcal{Y}_2$ is quasi-compact, there exists $m \in \mathbb{Z}$ such that for any $n \in \mathbb{Z}$ the functor $\pi_{\mathrm{dR},*}$, when restricted to* $\text{D-mod}(\mathcal{Y}_1)^{\geq n}$*, maps to* $\text{D-mod}(\mathcal{Y}_2)^{\geq n-m}$*, and as such commutes with filtered colimits.*

7.7. **Projection formula for the** $(\mathrm{dR}, *)$**-pushforward.**

[23] The proof uses the fact that for a morphism of affine DG schemes, the functor of !-pullback of D-modules has a finite cohomological amplitude.

7.7.1. In the situation of Sect. 7.4.1, let $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$ and $\mathcal{M}_2 \in \text{D-mod}(\mathcal{Y}_2)$ be two objects. We claim that there is always a morphism in one direction

$$\mathcal{M}_2 \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{M}_1) \to \pi_{\mathrm{dR},*}(\pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1). \tag{7.17}$$

Indeed, specifying such morphism amounts to a compatible family of maps

$$\mathcal{M}_2 \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{M}_1) \to (\pi \circ g)_{\mathrm{dR},*}\left(g^*_{\mathrm{dR}}\left(\pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1\right)\right)$$

for $(S, g) \in (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Y}_1,\mathrm{smooth}}$.

The required map arises from the map

$$\mathcal{M}_2 \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{M}_1) \to \mathcal{M}_2 \overset{!}{\otimes} (\pi \circ g)_{\mathrm{dR},*}(g^*_{\mathrm{dR}}(\mathcal{M}_1)) \simeq$$
$$\simeq (\pi \circ g)_{\mathrm{dR},*}\left((\pi \circ g)^!(\mathcal{M}_2) \overset{!}{\otimes} g^*_{\mathrm{dR}}(\mathcal{M}_1)\right) \simeq (\pi \circ g)_{\mathrm{dR},*}\left(g^!(\pi^!(\mathcal{M}_2)) \overset{!}{\otimes} g^*_{\mathrm{dR}}(\mathcal{M}_1)\right) \simeq$$
$$\simeq (\pi \circ g)_{\mathrm{dR},*}\left(g^*_{\mathrm{dR}}\left(\pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1\right)\right)$$

where the second arrow is furnished by Sect. 6.1.7, as the morphism $\pi \circ g$ is schematic and quasi-compact, and where the last arrow uses the isomorphism (6.3).

7.7.2. We give the following definitions:

Definition 7.7.3.

(a) *The triple* $(\mathcal{M}_1, \mathcal{M}_2, \pi)$ *satisfies the projection formula if the map* (7.17) *is an isomorphism.*

(b) *The pair* $(\mathcal{M}_2, \pi)$ *satisfies the projection formula if* (7.17) *is an isomorphism for any* $\mathcal{M}_1$.

(c) *The pair* $(\mathcal{M}_1, \pi)$ *satisfies the projection formula if* (7.17) *is an isomorphism for any* $\mathcal{M}_2$.

(d) *The map* π *satisfies the projection formula if* (7.17) *is an isomorphism for any* $\mathcal{M}_1$ *and* $\mathcal{M}_2$.

We also give the following definition:

Definition 7.7.4. *The morphism* π *strongly satisfies the projection formula if it satisfies base change and for every* $S_2 \in (\mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Y}_2}$, *the morphism*

$$\pi_S : S_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S_2$$

satisfies the projection formula.

It is easy to see that if π strongly satisfies the projection formula, then it satisfies the projection formula.

7.7.5. *Examples.*

(i) It is easy to see that if π is schematic and quasi-compact, then π strongly satisfies the projection formula.

(ii) In Theorem 10.2.4 we shall strengthen this to the assertion that any π which is *safe* also strongly satisfies the projection formula.

(iii) In Corollary 9.3.10, we will show that if π is a morphism between QCA stacks, and $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$ is *safe*, then $(\mathcal{M}_1, \pi)$ satisfies the projection formula.

(iv) Suppose that π is quasi-compact. Then for any $\mathcal{M}_i \in \text{D-mod}(\mathcal{Y}_i)^+$, the triple $(\mathcal{M}_1, \mathcal{M}_2, \pi)$ satisfies the projection formula. This follows in the same way as in Corollary 1.3.17(c), using the fact that for a quasi-compact algebraic stack $\mathcal{Y}$ the functor $\overset{!}{\otimes}$ on D-mod($\mathcal{Y}$) has a bounded cohomological amplitude.

7.7.6. *A counter-example.* It is easy to produce an example of how the projection formula fails when $\mathcal{M}_2$ is not compact. E.g., take $\mathcal{Y}_1 = B\mathbb{G}_m$, $\mathcal{Y}_2 = \mathrm{pt}$, $\mathcal{M}_1 = \underset{n\geq 0}{\oplus} k_{B\mathbb{G}_m}[2n]$ and $\mathcal{M}_2 = V$, where V is any infinite-dimensional vector space. Computing the two sides of the projection formula via Lemma 7.5.12, it is easy to check that the projection formula fails in this case.[24]

We shall now give an example of how the projection formula fails when $\mathcal{M}_2$ is compact.

Take $\mathcal{Y}_1 = \mathbb{A}^1 \times B\mathbb{G}_m$ and $\mathcal{Y}_2 = \mathbb{A}^1$, with the morphism π being the projection on the first factor:

$$\begin{CD} \mathbb{A}^1 \times B\mathbb{G}_m @>{p_{\mathbb{A}^1}\times \mathrm{id}_{B\mathbb{G}_m}}>> B\mathbb{G}_m \\ @V{\pi}VV @VV{p_{B\mathbb{G}_m}}V \\ \mathbb{A}^1 @>{p_{\mathbb{A}^1}}>> \mathrm{pt}\,. \end{CD}$$

We take $\mathcal{M}_2 := \mathbf{ind}^{\mathrm{left}}_{\text{D-mod}(\mathbb{A}^1)}(\mathcal{O}_{\mathbb{A}^1})$ and

$$\mathcal{M}_1 := (p_{\mathbb{A}^1} \times \mathrm{id}_{B\mathbb{G}_m})^! \left(\underset{n\geq 0}{\oplus} k_{B\mathbb{G}_m}[2n]\right).$$

We shall consider D-mod($\mathbb{A}^1$) in the "left" realization, and in particular, with the t-structure, for which the functor $\mathbf{oblv}^{\mathrm{left}}_{\text{D-mod}(\mathbb{A}^1)}$ is t-exact.

We calculate both $\pi_{\mathrm{dR},*}(\mathcal{M}_1)$ and $\pi_{\mathrm{dR},*}(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2))$ using Lemma 7.5.12. We have:

$$\pi_{\mathrm{dR},*}(\mathcal{M}_1) \simeq \underset{\underset{m}{\longleftarrow}}{lim}\ \pi_{\mathrm{dR},*} \circ (p_{\mathbb{A}^1} \times \mathrm{id}_{B\mathbb{G}_m})^! \left(\underset{m\geq n\geq 0}{\oplus} k_{B\mathbb{G}_m}[2n]\right)$$

and

$$\pi_{\mathrm{dR},*}(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)) \simeq$$
$$\simeq \underset{\underset{m}{\longleftarrow}}{lim}\ \pi_{\mathrm{dR},*}\left((p_{\mathbb{A}^1} \times \mathrm{id}_{B\mathbb{G}_m})^!(\underset{m\geq n\geq 0}{\oplus} k_{B\mathbb{G}_m}[2n]) \overset{!}{\otimes} \pi^!(\mathbf{ind}^{\mathrm{left}}_{\text{D-mod}(\mathbb{A}^1)}(\mathcal{O}_{\mathbb{A}^1}))\right).$$

By Proposition 7.6.8(a), for every m we have

$$\pi_{\mathrm{dR},*} \circ (p_{\mathbb{A}^1} \times \mathrm{id}_{B\mathbb{G}_m})^! \left(\underset{m\geq n\geq 0}{\oplus} k_{B\mathbb{G}_m}[2n]\right) \simeq$$
$$\simeq (p_{\mathbb{A}^1})^! \left(\Gamma_{\mathrm{dR}}(B\mathbb{G}_m, (\underset{m\geq n\geq 0}{\oplus} k_{B\mathbb{G}_m}[2n]))\right) \simeq$$
$$\simeq (p_{\mathbb{A}^1})^! \left(\underset{m\geq n\geq 0, l\geq 0}{\oplus} k[2(n-l)]\right) \simeq \underset{m\geq n\geq 0, l\geq 0}{\oplus} \mathcal{O}_{\mathbb{A}^1}[2(n-l)].$$

In particular, the 0th cohomology of $\pi_{\mathrm{dR},*}(\mathcal{M}_1)$ identifies with

$$\underset{m\geq 0}{\Pi} \mathcal{O}_{\mathbb{A}^1},$$

the countable product of copies of $\mathcal{O}_{\mathbb{A}^1} \in \text{D-mod}^{\mathrm{left}}(\mathbb{A}^1)$.

[24]Doing this exercise makes it easier to read similar but more lengthy computations below.

Since $\mathbf{ind}^{\text{left}}_{\text{D-mod}(\mathbb{A}^1)}(\mathcal{O}_{\mathbb{A}^1})$ is flat as an $\mathcal{O}_{\mathbb{A}^1}$-module, we obtain that the 0th cohomology of $\pi_{\text{dR},*}(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2$ identifies with

$$\left(\prod_{m\geq 0} \mathcal{O}_{\mathbb{A}^1}\right) \otimes \mathbf{ind}^{\text{left}}_{\text{D-mod}(\mathbb{A}^1)}(\mathcal{O}_{\mathbb{A}^1})$$

(we note that in the "left" realization, the tensor product $\overset{!}{\otimes}$ corresponds to the usual tensor product $\otimes$ at the level of the underlying $\mathcal{O}$-modules).

Note that the forgetful functor

$$\Gamma(\mathbb{A}^1, -) \circ \mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathbb{A}^1)} : \text{D-mod}(\mathbb{A}^1) \to \text{Vect}$$

commutes with limits, since it admits a left adjoint. Hence, we obtain that the 0th cohomology of

$$\Gamma\left(\mathbb{A}^1, \mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathbb{A}^1)}(\pi_{\text{dR},*}(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2)\right)$$

identifies with

$$\left(\prod_{m\geq 0} k[t]\right) \underset{k[t]}{\otimes} k[t,\partial_t] \simeq \left(\prod_{m\geq 0} k[t]\right) \otimes V,$$

where V is a vector space such that $k[t,\partial_t] \simeq k[t] \otimes V$ as a $k[t]$-module. The key point is that V is infinite-dimensional.

By Sect. 7.7.5(iv),

$$\pi_{\text{dR},*}\left((p_{\mathbb{A}^1} \times \text{id}_{B\mathbb{G}_m})^!(\underset{m\leq n\geq 0}{\oplus} k_{B\mathbb{G}_m}[2n]) \overset{!}{\otimes} \pi^!(\mathcal{M}_2)\right) \simeq$$

$$\simeq (p_{\mathbb{A}^1})^!(\underset{m\geq n\geq 0, l\geq 0}{\oplus} k[2(n-l)]) \overset{!}{\otimes} \mathbf{ind}_{\text{D-mod}(\mathbb{A}^1)}(\mathcal{O}_{\mathbb{A}^1}) \simeq$$

$$\simeq \left(\underset{m\geq n\geq 0, l\geq 0}{\oplus} \mathcal{O}_{\mathbb{A}^1}[2(n-l)]\right) \otimes \mathbf{ind}_{\text{D-mod}(\mathbb{A}^1)}(\mathcal{O}_{\mathbb{A}^1}).$$

Hence, the 0th cohomology of

$$\Gamma\left(\mathbb{A}^1, \mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathbb{A}^1)}(\pi_{\text{dR},*}(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)))\right)$$

identifies with

$$\prod_{m\geq 0} k[t,\partial_t] \simeq \prod_{m\geq 0} (k[t] \otimes V).$$

Finally, the canonical map

$$\left(\prod_{m\geq 0} k[t]\right) \otimes V \to \prod_{m\geq 0} (k[t] \otimes V)$$

is *not* an isomorphism because V is infinite-dimensional.

7.8. **Proofs of properties of the** $(\text{dR},*)$**-pushforward.**

7.8.1. First, we are now going to prove the following assertion, which has multiple consequences:

Lemma 7.8.2. *For a map of algebraic stacks $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$, and $\mathcal{M}_2 \in \text{D-mod}_{\text{coh}}(\mathcal{Y}_2)$ and any $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$, the map*

$$\Gamma_{\text{dR}}(\mathcal{Y}_2, \mathcal{M}_2 \overset{!}{\otimes} \pi_{\text{dR},*}(\mathcal{M}_1)) \to \Gamma_{\text{dR}}(\mathcal{Y}_2, \pi_{\text{dR},*}(\pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1)),$$

induced by (7.17), *is an isomorphism.*

Remark 7.8.3. Note, however, that in the situation of the Lemma 7.8.2, the map (7.17) itself does not have to be an isomorphism, see example in Sect. 7.7.6 above.

Proof of Lemma 7.8.2. Note that the assumption that $\mathcal{M}_2 \in \text{D-mod}_{\text{coh}}(\mathcal{Y}_2)$ implies that the functor

$$\mathcal{M}_2' \mapsto \Gamma_{\text{dR}}(\mathcal{Y}_2, \mathcal{M}_2' \overset{!}{\otimes} \mathcal{M}_2) : \text{D-mod}(\mathcal{Y}_2) \to \text{Vect}$$

commutes with limits. Indeed, this is because the above functor identifies with

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{Y}_2)}(\mathbb{D}^{\text{Verdier}}_{\mathcal{Y}_2}(\mathcal{M}_2), -),$$

by Lemma 7.3.5.

Applying the definition of $\pi_{\text{dR},*}$, we obtain that $\Gamma_{\text{dR}}\left(\mathcal{Y}_2, \pi_{\text{dR},*}(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2\right)$ identifies with the limit over $(S, g) \in \text{DGSch}_{/\mathcal{Y}_1,\text{smooth}}$ of

$$\Gamma_{\text{dR}}\left(\mathcal{Y}_2, (\pi \circ g)_{\text{dR},*}(g^*_{\text{dR}}(\mathcal{M}_1)) \overset{!}{\otimes} \mathcal{M}_2\right).$$

Since the morphism $\pi \circ g$ is schematic and quasi-compact, the projection formula (6.2) is applicable, and the latter expression can be rewritten as

$$\Gamma_{\text{dR}}\left(\mathcal{Y}_2, (\pi \circ g)_{\text{dR},*}\left(g^*_{\text{dR}}(\mathcal{M}_1) \overset{!}{\otimes} (\pi \circ g)^!(\mathcal{M}_2)\right)\right),$$

and by Lemma 7.5.6 further as

$$\Gamma_{\text{dR}}\left(S, g^*_{\text{dR}}(\mathcal{M}_1) \overset{!}{\otimes} (\pi \circ g)^!(\mathcal{M}_2)\right) \simeq \Gamma_{\text{dR}}\left(S, g^*_{\text{dR}}(\mathcal{M}_1) \overset{!}{\otimes} g^!(\pi^!(\mathcal{M}_2))\right).$$

Applying the isomorphism of (6.3), we rewrite the latter as

$$\Gamma_{\text{dR}}\left(S, g^*_{\text{dR}}\left(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)\right)\right).$$

Now, the resulting limit over (S, g) is isomorphic to $\Gamma_{\text{dR}}(\mathcal{Y}_1, \mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2))$ by definition. $\square$

Note that Lemma 7.8.2 gives the following, somewhat more explicit characterization of the $(\text{dR}, *)$-pushforward functor:

Corollary 7.8.4. *For $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$, $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$ and $\mathcal{M}_2 \in \text{D-mod}_{\text{coh}}(\mathcal{Y}_2)$, we have a canonical isomorphism*

$$\mathcal{M}aps(\mathcal{M}_2, \pi_{\text{dR},*}(\mathcal{M}_1)) \simeq \Gamma_{\text{dR}}(\mathcal{Y}_1, \pi^!(\mathbb{D}^{\text{Verdier}}_{\mathcal{Y}_2}(\mathcal{M}_2)) \overset{!}{\otimes} \mathcal{M}_1).$$

Proof. Using Lemma 7.3.5, we rewrite the left-hand side as

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_2, \mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}_2}(\mathcal{M}_2) \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{M}_1)),$$

and further, using Lemma 7.8.2, as

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_2, \pi_{\mathrm{dR},*}(\pi^!(\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}_2}(\mathcal{M}_2)) \overset{!}{\otimes} \mathcal{M}_1)),$$

and finally as

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_1, \pi^!(\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}_2}(\mathcal{M}_2)) \overset{!}{\otimes} \mathcal{M}_1)$$

using Lemma 7.5.6.

□

7.8.5. *Proof of Proposition 7.5.7.* It is easy to see that for an algebraic stack $\mathcal{Y}$, the category $\text{D-mod}(\mathcal{Y})$ is generated by its subcategory $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$. Hence, using Lemma 7.3.5, we obtain that it is enough to show that for $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$ and $\mathcal{M}_3 \in \text{D-mod}_{\mathrm{coh}}(\mathcal{Y}_3)$, the map

$$\Gamma_{\mathrm{dR}}\left(\mathcal{Y}_3, \mathcal{M}_3 \overset{!}{\otimes} (\phi_{\mathrm{dR},*} \circ \pi_{\mathrm{dR},*}(\mathcal{M}_1))\right) \to \Gamma_{\mathrm{dR}}\left(\mathcal{Y}_3, \mathcal{M}_3 \overset{!}{\otimes} (\phi \circ \pi)_{\mathrm{dR},*}(\mathcal{M}_1)\right)$$

is an isomorphism.

Applying Lemmas 7.8.2 and 7.5.6, we rewrite

$$\begin{aligned}\Gamma_{\mathrm{dR}}\left(\mathcal{Y}_3, \mathcal{M}_3 \overset{!}{\otimes} (\phi_{\mathrm{dR},*} \circ \pi_{\mathrm{dR},*}(\mathcal{M}_1))\right) \simeq \Gamma_{\mathrm{dR}}\left(\mathcal{Y}_3, \phi_{\mathrm{dR},*}(\phi^!(\mathcal{M}_3) \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{M}_1))\right) \simeq \\ \simeq \Gamma_{\mathrm{dR}}\left(\mathcal{Y}_2, \phi^!(\mathcal{M}_3) \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{M}_1)\right)\end{aligned} \tag{7.18}$$

and

$$\begin{aligned}\Gamma_{\mathrm{dR}}\left(\mathcal{Y}_3, \mathcal{M}_3 \overset{!}{\otimes} (\phi \circ \pi)_{\mathrm{dR},*}(\mathcal{M}_1)\right) \simeq \Gamma_{\mathrm{dR}}\left(\mathcal{Y}_3, (\phi \circ \pi)_{\mathrm{dR},*}((\phi \circ \pi)^!(\mathcal{M}_3) \overset{!}{\otimes} \mathcal{M}_1)\right) \simeq \\ \simeq \Gamma_{\mathrm{dR}}\left(\mathcal{Y}_1, (\phi \circ \pi)^!(\mathcal{M}_3) \overset{!}{\otimes} \mathcal{M}_1\right).\end{aligned} \tag{7.19}$$

Since π is schematic and quasi-compact, the projection formula is applicable, and we obtain

$$\Gamma_{\mathrm{dR}}\left(\mathcal{Y}_2, \phi^!(\mathcal{M}_3) \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{M}_1)\right) \simeq \Gamma_{\mathrm{dR}}\left(\mathcal{Y}_2, \pi_{\mathrm{dR},*}(\pi^! \circ \phi^!(\mathcal{M}_3) \overset{!}{\otimes} \mathcal{M}_1)\right). \tag{7.20}$$

Hence, using Lemma 7.5.6, we obtain that the expression in (7.18) is also isomorphic to

$$\Gamma_{\mathrm{dR}}\left(\mathcal{Y}_1, (\phi \circ \pi)^!(\mathcal{M}_3) \overset{!}{\otimes} \mathcal{M}_1\right),$$

as required.

□

7.8.6. Note that the only non-tautological point of the proof of Proposition 7.5.7 is the isomorphism (7.20).

Hence, more generally, we obtain that the map (7.14) is an isomorphism in the following situations:

(i) When π satisfies the projection formula.

(ii) When $\phi^!$ sends $\text{D-mod}_{\text{coh}}(\mathcal{Y}_3)$ to $\text{D-mod}_{\text{coh}}(\mathcal{Y}_2)$ (this is due to Lemma 7.8.2). This happens, e.g., when ϕ is smooth.

(iii) When $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)^+$ (this is due to Sect. 7.7.5(iv)).

7.8.7. The natural transformation (7.14) fails to be an isomorphism in the following example.

We take $\mathcal{Y}_1 = B\mathbb{G}_m$, $\mathcal{Y}_2 = \text{pt}$ and $\mathcal{Y}_3 = \mathbb{A}^1$, where ϕ is the inclusion of 0 into $\mathbb{A}^1$. We take $\mathcal{M}_1 \in \text{D-mod}(B\mathbb{G}_m)$ equal to $\underset{n\geq 0}{\oplus} k_{B\mathbb{G}_m}[2n]$. We claim that

$$\mathcal{M}_3 := (\phi\circ\pi)_{\text{dR},*}(\mathcal{M}_1) \in \text{D-mod}(\mathbb{A}^1)$$

is *not* supported at 0. Indeed, using the fact that the functors

$$\Gamma(\mathbb{A}^1,-) : \text{QCoh}(\mathbb{A}^1) \to \text{Vect} \text{ and } \mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathbb{A}^1)} : \text{D-mod}(\mathbb{A}^1) \to \text{QCoh}(\mathbb{A}^1)$$

commute with limits, we calculate $\Gamma(\mathbb{A}^1, \mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathbb{A}^1)}(\mathcal{M}_3))$ via Lemma 7.5.12.

Note that

$$(\phi\circ\pi)_{\text{dR},*}(k_{B\mathbb{G}_m}) \simeq \underset{m\geq 0}{\oplus} \delta[-2m],$$

where δ is the δ-function at $0 \in \mathbb{A}^1$. We obtain that $H^0\left(\Gamma(\mathbb{A}^1, \mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathbb{A}^1)}(\mathcal{M}_3))\right)$ is the *product* of $\mathbb{N}$-many copies of $\Gamma(\mathbb{A}^1, \mathbf{oblv}^{\text{left}}_{\text{D-mod}(\mathbb{A}^1)}(\delta))$. In particular, the generator $t \in \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}_1})$ acts on it non-nilpotently.

7.8.8. *Proof of Lemma 7.5.2.* As in the proof of Proposition 7.5.7, it suffices to show that for $\mathcal{M}_2 \in \text{D-mod}_{\text{coh}}(\mathcal{Y}_2)$, the map

$$\Gamma_{\text{dR}}(\mathcal{Y}_2, \mathcal{M}_2 \overset{!}{\otimes} \pi_{\text{dR},*}(\mathcal{M}_1)) \to \Gamma_{\text{dR}}\left(\mathcal{Y}_2, \mathcal{M}_2 \overset{!}{\otimes}\left(\underset{a\in A^{\text{op}}}{\underleftarrow{lim}} (\pi\circ g_\alpha)_{\text{dR},*}\circ (g_\alpha)^*_{\text{dR}}(\mathcal{M}_1)\right)\right)$$

is an isomorphism.

As in the proof of Lemma 7.8.2, we have:

$$\Gamma_{\text{dR}}\left(\mathcal{Y}_2, \mathcal{M}_2 \overset{!}{\otimes}\left(\underset{a\in A^{\text{op}}}{\underleftarrow{lim}} (\pi\circ g_\alpha)_{\text{dR},*}\circ (g_\alpha)^*_{\text{dR}}(\mathcal{M}_1)\right)\right) \simeq$$
$$\simeq \underset{a\in A^{\text{op}}}{\underleftarrow{lim}} \Gamma_{\text{dR}}\left(S_\alpha, (g_\alpha)^*_{\text{dR}}(\pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1)\right) \simeq$$
$$\simeq \underset{a\in A^{\text{op}}}{\underleftarrow{lim}} \mathcal{M}aps_{\text{D-mod}(S_\alpha)}\left(k_{S_\alpha}, (g_\alpha)^*_{\text{dR}}(\pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1)\right)$$

However, by the assumption on A, the latter expression is isomorphic to

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{Y}_1)}(k_{\mathcal{Y}_1}, \pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1) \simeq \Gamma_{\text{dR}}(\mathcal{Y}_1, \pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1),$$

which is isomorphic to

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_2, \mathcal{M}_2 \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{M}_1)),$$

by Lemmas 7.8.2 and 7.5.6

□

7.9. **Proof of Proposition 7.1.6.**

7.9.1. First, we are going to construct a map in one direction:

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})) \to \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, \mathcal{F}). \tag{7.21}$$

By definition, the left-hand side and the right-hand side are the limits over

$$(S, g) \in (\mathrm{DGSch}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}$$

of

$$\Gamma_{\mathrm{dR}}(S, g^*_{\mathrm{dR}} \circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})) \text{ and } \Gamma^{\mathrm{IndCoh}}(S, g^{\mathrm{IndCoh},*}(\mathcal{F})),$$

respectively.

We rewrite

$$\Gamma^{\mathrm{IndCoh}}(S, g^{\mathrm{IndCoh},*}(\mathcal{F})) \simeq \Gamma_{\mathrm{dR}}(S, \mathbf{ind}_{\text{D-mod}(S)} \circ g^{\mathrm{IndCoh},*}(\mathcal{F})).$$

We claim that there is a canonical map

$$g^*_{\mathrm{dR}} \circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}) \to \mathbf{ind}_{\text{D-mod}(S)} \circ g^{\mathrm{IndCoh},*}(\mathcal{F})$$

that functorially depends on (S, g). The map in question arises by the $(g^*_{\mathrm{dR}}, g_{\mathrm{dR},*})$ adjunction from the isomorphism of Proposition 6.5.7.

Thus, we obtain a compatible system of maps

$$\Gamma_{\mathrm{dR}}(S, g^*_{\mathrm{dR}} \circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})) \to \Gamma^{\mathrm{IndCoh}}(S, g^{\mathrm{IndCoh},*}(\mathcal{F})), \tag{7.22}$$

giving rise to the desired map (7.21).

Note, however, that the individual maps in (7.22) are *not* isomorhisms.

7.9.2. The following property of the map (7.21) follows from the construction. Let $\pi : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be a schematic and quasi-compact map.

Then for $\widetilde{\mathcal{F}} \in \mathrm{IndCoh}(\widetilde{\mathcal{Y}})$ the following diagram commutes:

$$\begin{tikzcd}
\Gamma_{\mathrm{dR}}(\widetilde{\mathcal{Y}}, \mathbf{ind}_{\text{D-mod}(\widetilde{\mathcal{Y}})}(\widetilde{\mathcal{F}})) \arrow[r, "(7.21)"] \arrow[d, "\text{Lemma 7.5.6}"', "\sim"] & \Gamma^{\mathrm{IndCoh}}(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{F}}) \arrow[dd, "\sim"] \\
\Gamma_{\mathrm{dR}}(\mathcal{Y}, \pi_{\mathrm{dR},*} \circ \mathbf{ind}_{\text{D-mod}(\widetilde{\mathcal{Y}})}(\widetilde{\mathcal{F}})) \arrow[d, "\text{Proposition 6.5.7}"', "\sim"] & \\
\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ \pi^{\mathrm{IndCoh}}_*(\widetilde{\mathcal{F}})) \arrow[r, "(7.21)"] & \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, \pi^{\mathrm{IndCoh}}_*(\widetilde{\mathcal{F}})).
\end{tikzcd}$$

7.9.3. *Two reduction steps.* We note that for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y})$, the maps

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})) \to \underset{\underset{n}{\longleftarrow}}{lim}\, \Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\tau^{\geq -n}(\mathcal{F})))$$

and

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, \mathcal{F}) \to \underset{\underset{n}{\longleftarrow}}{lim}\, \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, \tau^{\geq -n}(\mathcal{F}))$$

are both isomorphisms.

Another way to phrase this is that both functors are right Kan extensions of their restrictions to $\mathrm{IndCoh}(\mathcal{Y})^{+}$.

Hence, in order to show that (7.21) is an isomorphism, it is enough to do so for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y})^{+}$. Passing to a Zariski cover, we may assume that $\mathcal{Y}$ is quasi-compact.

7.9.4. Choose a smooth cover $g : Z \to \mathcal{Y}$, where $Z \in \mathrm{DGSch}_{\mathrm{aft}}$, and consider its Čech nerve $Z^{\bullet}/\mathcal{Y}$. Let g^i denote the corresponding map $Z^i/\mathcal{Y} \to \mathcal{Y}$.

Consider the resulting map

$$\mathcal{F} \to \mathrm{Tot}\left((g^i)_*^{\mathrm{IndCoh}} \circ (g^i)^{\mathrm{IndCoh},*}(\mathcal{F})\right).$$

It is an isomorphism in $\mathrm{IndCoh}(\mathcal{Y})$, because the terms are uniformly bounded below, and the corresponding map

$$\Psi_{\mathcal{Y}}(\mathcal{F}) \to \Psi_{\mathcal{Y}}\left(\mathrm{Tot}\left((g^i)_*^{\mathrm{IndCoh}} \circ (g^i)^{\mathrm{IndCoh},*}(\mathcal{F})\right)\right) \simeq \mathrm{Tot}\left((g^i)_* \circ (g^i)^*(\Psi_{\mathcal{Y}}(\mathcal{F}))\right)$$

is an isomorphism in $\mathrm{QCoh}(\mathcal{Y})$.

Since $\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}-) \simeq \Gamma(\mathcal{Y}, -) \circ \Psi_{\mathcal{Y}}$, the map

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, \mathcal{F}) \to \mathrm{Tot}\left(\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, (g^i)_*^{\mathrm{IndCoh}} \circ (g^i)^{\mathrm{IndCoh},*}(\mathcal{F}))\right) \simeq$$
$$\simeq \mathrm{Tot}\left(\Gamma^{\mathrm{IndCoh}}(Z^i/\mathcal{Y}, (g^i)^{\mathrm{IndCoh},*}(\mathcal{F}))\right)$$

is also an isomorphism.

7.9.5. We claim that the map

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}) \to \mathrm{Tot}\left(\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ (g^i)_*^{\mathrm{IndCoh}} \circ (g^i)^{\mathrm{IndCoh},*}(\mathcal{F})\right) \overset{\text{Proposition 6.5.7}}{\simeq}$$
$$\simeq \mathrm{Tot}\left((g^i)_{\mathrm{dR},*} \circ \mathbf{ind}_{\text{D-mod}(Z^i/\mathcal{Y})} \circ (g^i)^{\mathrm{IndCoh},*}(\mathcal{F})\right)$$

is an isomorphism.

This is obtained as in Corollary 1.3.17(c) using the fact that the functor $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ is of bounded cohomological amplitude.

Note also that the functor $\Gamma_{\mathrm{dR}}(\mathcal{Y}, -)$ commutes with limits. Indeed, the functor in question is given by $\mathcal{M}aps_{\text{D-mod}(\mathcal{Y})}(k_{\mathcal{Y}}, -)$.

Hence, we obtain that the natural map

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})) \to \mathrm{Tot}\left(\Gamma_{\mathrm{dR}}\left(\mathcal{Y}, (g^i)_{\mathrm{dR},*} \circ \mathbf{ind}_{\text{D-mod}(Z^i/\mathcal{Y})} \circ (g^i)^{\mathrm{IndCoh},*}(\mathcal{F})\right)\right) \simeq$$
$$\overset{\text{Lemma 7.5.6}}{\simeq} \mathrm{Tot}\left(\Gamma_{\mathrm{dR}}\left(Z^i/\mathcal{Y}, \mathbf{ind}_{\text{D-mod}(Z^i/\mathcal{Y})} \circ (g^i)^{\mathrm{IndCoh},*}(\mathcal{F})\right)\right)$$

is an isomorphism.

7.9.6. Now, it follows from Sect. 7.9.2 that the map in (7.21) fits into the commutative diagram

$$\begin{array}{ccc}
\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})) & \longrightarrow & \mathrm{Tot}\left(\Gamma_{\mathrm{dR}}\left(Z^i/\mathcal{Y}, \mathbf{ind}_{\text{D-mod}(Z^i/\mathcal{Y})} \circ (g^i)^{\mathrm{IndCoh},*}(\mathcal{F})\right)\right) \\
\downarrow & & \downarrow \\
\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, \mathcal{F}) & \longrightarrow & \mathrm{Tot}\left(\Gamma^{\mathrm{IndCoh}}(Z^i/\mathcal{Y}, (g^i)^{\mathrm{IndCoh},*}(\mathcal{F}))\right),
\end{array}$$

where the right vertical arrow is a co-simplicial isomorphism coming from

$$\Gamma_{\mathrm{dR}}\left(Z^i/\mathcal{Y}, \mathbf{ind}_{\text{D-mod}(Z^i/\mathcal{Y})}(-)\right) \simeq \Gamma^{\mathrm{IndCoh}}(Z^i/\mathcal{Y}, -).$$

This implies that the left vertical arrow is an isomorphism, as required.

□

8. Compact generation of D-mod($\mathcal{Y}$)

In this section we will finally prove the result that caused out to write this paper: that for a QCA algebraic stack $\mathcal{Y}$, the category D-mod($\mathcal{Y}$) is compactly generated. After all the preparations we have made, the proof will be extremely short. In Sect. 8.3 we shall establish some additional favorable properties of the category D-mod($\mathcal{Y}$).

Throughout this section, we will assume that unless specified otherwise, all our (pre)stacks are QCA algebraic stacks in the sense of Definition 1.1.8 (in particular, they are quasi-compact).

8.1. Proof of compact generation.

Theorem 8.1.1. *The category* D-mod($\mathcal{Y}$) *is compactly generated. More precisely, objects of* D-mod($\mathcal{Y}$) *of the form*

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}), \quad \mathcal{F} \in \mathrm{Coh}(\mathcal{Y}) \tag{8.1}$$

are compact and generate D-mod($\mathcal{Y}$).

Proof. (i) By Proposition 3.4.2, the objects of Coh($\mathcal{Y}$) are compact in IndCoh($\mathcal{Y}$).[25] Since $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ is the left adjoint of a functor that commutes with colimits, it sends compact objects to compact ones. So objects of the form (8.1) are compact.

(ii) By Proposition 3.5.1, Coh($\mathcal{Y}$) generates IndCoh($\mathcal{Y}$). So it remains to show that the essential image of $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ generates D-mod($\mathcal{Y}$). This follows from the fact that the functor $\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}$ is conservative. □

Remark 8.1.2. Note that, unlike the case of DG schemes, the subcategory

$$\text{D-mod}(\mathcal{Y})^c \subset \text{D-mod}(\mathcal{Y})$$

is *not* preserved by the truncation functors. We note that this is also the case for the category QCoh($-$) on non-regular schemes. By contrast, IndCoh$(-)^c$ on schemes and QCA algebraic stacks is compatible with the t-structure.

8.2. **Variant of the proof of Theorem 8.1.1.** For the reader who prefers to avoid the (potentially unfamiliar) category IndCoh($\mathcal{Y}$), below we give an alternative argument, which does not use IndCoh($\mathcal{Y}$) explicitly. Since the assertion is about categorical properties of D-mod($\mathcal{Y}$), we may assume that $\mathcal{Y}$ is a classical stack (rather than a DG stack). [26]

[25] Recall that the proof of this fact is based on formula (3.21) and Theorem 1.4.2.

[26] The same proof is applicable when $\mathcal{Y}$ is an eventually coconnective QCA stack.

8.2.1. Recall the pair of adjoint functors

$$'\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} : \mathrm{QCoh}(\mathcal{Y}) \rightleftarrows \text{D-mod}(\mathcal{Y}) : '\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})},$$

see Sect. 6.3.21, and recall that $'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}$ is conservative.

Hence, by Corollary 1.4.11, in order to prove Theorem 8.1.1, it is sufficient to show that the functor $'\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ sends $\mathrm{Coh}(\mathcal{Y}) \subset \mathrm{QCoh}(\mathcal{Y})$ to $\text{D-mod}(\mathcal{Y})^c$.

I.e., we need to show that for $\mathcal{F} \in \mathrm{Coh}(\mathcal{Y})$, the functor $\text{D-mod}(\mathcal{Y}) \to \mathrm{Vect}$ defined by

$$\mathcal{M} \mapsto \mathcal{M}aps_{\text{D-mod}(\mathcal{Y})}('\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}), \mathcal{M}), \qquad \mathcal{M} \in \text{D-mod}(\mathcal{Y})$$

is continuous.

The idea of the proof is the same as that of Proposition 3.4.2: namely, we represent the above functor as a composition of a continuous functor

$$\text{D-mod}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}), \quad \mathcal{M} \mapsto \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}, '\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})) \tag{8.2}$$

and the functor $\Gamma(\mathcal{Y}, -) : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}$, which is also continuous by Theorem 1.4.2.

The functor (8.2) is the functor of "internal Hom" from a coherent sheaf to a D-module. The content of the proof is to show that the latter is well-defined and has the expected properties.

8.2.2. We rewrite the expression $\mathcal{M}aps_{\text{D-mod}(\mathcal{Y})}('\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}), \mathcal{M})$ as

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}, '\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})).$$

We introduce the object

$$\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}, '\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})) \in \mathrm{QCoh}(\mathcal{Y})$$

as follows.

Let $\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{smooth}}$ denote the full subcategory of the category of affine schemes over $\mathcal{Y}$, where we restrict objects $(S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}, g : S \to \mathcal{Y})$ to those for which g is smooth. We restrict 1-morphisms to those $f : S' \to S$ for which f is smooth.

For

$$(S, g) \in (\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}},$$

we set

$$\begin{aligned} \Gamma\left(S, g^*(\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}, '\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})))\right) := \\ = \mathcal{M}aps_{\mathrm{QCoh}(S)}\left(g^!(\mathcal{F}), '\mathbf{oblv}_{\text{D-mod}(S)}(g^!(\mathcal{M}))\right). \end{aligned} \tag{8.3}$$

Here $g^!$ is well-defined as a functor $\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(S)$ since g is smooth.

8.2.3. We claim:

Lemma 8.2.4. *For $f : S' \to S$ in $\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{smooth}}$, the natural map*

$$f^*\left(g^*(\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}, '\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})))\right) \to (g \circ f)^*\left(\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}, '\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))\right)$$

is an isomorphism.

The proof will be given in Sect. 8.2.7. The above lemma ensures that the assignment

$$(S,g)\mapsto g^*\left(\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))\right)$$

indeed defines an object of $\mathrm{QCoh}(\mathcal{Y})$. We denote it by $\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))$.

Since $g^!$ is isomorphic to g^* up to a twist by a line bundle, we have:

$$\begin{aligned}
&\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))\simeq\\
&\simeq \underset{(S,g)\in\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{smooth}}}{\underset{\longleftarrow}{lim}} \mathcal{M}aps_{\mathrm{QCoh}(S)}\left(g^*(\mathcal{F}),g^*({}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))\right)\simeq\\
&\simeq \underset{(S,g)\in\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{smooth}}}{\underset{\longleftarrow}{lim}} \mathcal{M}aps_{\mathrm{QCoh}(S)}\left(g^!(\mathcal{F}),g^!({}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))\right)=\\
&= \underset{(S,g)\in\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{smooth}}}{\underset{\longleftarrow}{lim}} \Gamma\left(S,g^*(\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})))\right)=\\
&= \Gamma\left(\mathcal{Y},\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))\right).
\end{aligned}$$

Applying Theorem 1.4.2(i), we obtain that it suffices to show that the functor

$$\mathcal{M}\mapsto \underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))$$

commutes with colimits in $\mathcal{M}$.

Remark 8.2.5. A similar manipulation shows that for $\mathcal{F}_1\in\mathrm{QCoh}(\mathcal{Y})$,

$$\begin{aligned}
&\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}_1\otimes\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))\simeq\\
&\simeq \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}\left(\mathcal{F}_1,\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))\right);
\end{aligned}$$

in other words, $\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}))$ *is* the internal Hom object

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})).$$

8.2.6. Now let us prove continuity of the functor (8.2). Since for $(S,g)\in\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{smooth}}$, the functor g^* is continuous, it suffices to show that for every (S,g) as above, the functor

$$\mathcal{M}\mapsto g^*(\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})))$$

is continuous.

We rewrite

$$\begin{aligned}
&\Gamma\left(S,g^*(\underline{\mathrm{Hom}}'_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F},{}'\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})))\right)\simeq\\
&\simeq \mathcal{M}aps_{\text{D-mod}(S)}\left({}'\mathbf{ind}_{\text{D-mod}(S)}(g^!(\mathcal{F})),g^!(\mathcal{M})\right).
\end{aligned}\tag{8.4}$$

Now, $g^!(\mathcal{F})\in\mathrm{Coh}(S)$, and since S is a scheme, the functor ${}'\mathbf{ind}_{\text{D-mod}(S)}$ is known to send $\mathrm{Coh}(S)$ to $\text{D-mod}(S)^c$. This implies that the right-hand side in (8.4) commutes with colimits in $\mathcal{M}$.

□

8.2.7. *Proof of Lemma 8.2.4.* This will be parallel to the proof of Lemma 3.4.4.

Let $f : S' \to S$ be a smooth map between affine schemes. Let $\mathcal{F}$ be an object of $\mathrm{Coh}(S)$, and $\mathcal{M}$ an object of $\mathrm{D\text{-}mod}(S)$. We claim that the natural map

$$H^0\left(f^*\left(\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S)}(\mathcal{F}, {}'\mathbf{oblv}_{\mathrm{D\text{-}mod}(S)}(\mathcal{M}))\right)\right) \to \\ \to H^0\left(\underline{\mathrm{Hom}}_{\mathrm{QCoh}(S')}\left(f^!(\mathcal{F}), {}'\mathbf{oblv}_{\mathrm{D\text{-}mod}(S)}(f^!(\mathcal{M}))\right)\right). \tag{8.5}$$

is an isomorphism.

Note that the assumption that f is smooth and the fact that the categories $\mathrm{D\text{-}mod}(S)$ and $\mathrm{D\text{-}mod}(S')$ are of finite cohomological dimension, imply that both sides in (8.5) will remain unchanged if we replace $\mathcal{M}$ by $\tau^{\geq -n}(\mathcal{M})$ for $n \gg 0$.

Note also that (8.5) is evidently an isomorphism if $\mathcal{F} \in \mathrm{QCoh}(S)^c = \mathrm{QCoh}(S)^{\mathrm{perf}}$. Now replace $\mathcal{F}$ by $\mathcal{F}_1$, where $\mathcal{F}_1 \in \mathrm{QCoh}(S)^c$ is equipped with a map to $\mathcal{F}$, such that

$$\mathrm{Cone}(\mathcal{F}_1 \to \mathcal{F}) \in \mathrm{QCoh}(S)^{\leq -n}$$

with $n \gg 0$.

$\square$

8.3. **Some corollaries of Theorem 8.1.1.**

8.3.1. First we claim:

Corollary 8.3.2. $\mathrm{D\text{-}mod}(\mathcal{Y})^c$ *is Karoubi-generated by objects of the form* $\mathbf{ind}_{\mathrm{D\text{-}mod}(\mathcal{Y})}(\mathcal{F})$, $\mathcal{F} \in \mathrm{Coh}(\mathcal{Y})$.

Recall that for a cocomplete DG category $\mathbf{C}$ and its not necessarily cocomplete DG subcategories $\mathbf{C}'_0 \subset \mathbf{C}'$, ones says that a subcategory $\mathbf{C}'_0$ to Karoubi-generates $\mathbf{C}'$ if the latter is the smallest among DG subcategories of $\mathbf{C}$ that contain $\mathbf{C}'_0$ and are closed under direct summands. This is a condition on corresponding homotopy categories (i.e., it is insensitive to the ∞-category structure).

Proof. This follows from Sect. 0.6.7. $\square$

8.3.3. As yet another corollary of Theorem 8.1.1, we obtain:

Corollary 8.3.4. *Let* $\mathcal{Y}$ *be a QCA stack and* $\mathcal{Y}'$ any *prestack. Then the natural functor*

$$\mathrm{D\text{-}mod}(\mathcal{Y}) \otimes \mathrm{D\text{-}mod}(\mathcal{Y}') \to \mathrm{D\text{-}mod}(\mathcal{Y} \times \mathcal{Y}')$$

is an equivalence.

Proof. The proof repeats verbatim that of Corollary 4.2.3. It applies to *any* prestack $\mathcal{Y}$, for which the category $\mathrm{D\text{-}mod}(\mathcal{Y})$ is dualizable.

$\square$

8.4. **Verdier duality on a QCA stack.**

8.4.1. In Sect. 7.3.4 we defined an involutive anti self-equivalence

$$\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}} : (\text{D-mod}_{\mathrm{coh}}(\mathcal{Y}))^{\mathrm{op}} \to \text{D-mod}_{\mathrm{coh}}(\mathcal{Y}).$$

Corollary 8.4.2. *This functor induces an involutive anti self-equivalence*

$$\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}} : (\text{D-mod}(\mathcal{Y})^c)^{\mathrm{op}} \xrightarrow{\sim} \text{D-mod}(\mathcal{Y})^c \tag{8.6}$$

Proof. The nontrivial statement to prove is that $\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}}$ preserves $\text{D-mod}(\mathcal{Y})^c$. By Corollary 8.3.2, it suffices to show that $\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}}$ preserves $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathrm{Coh}(\mathcal{Y}))$. The latter follows from Corollary 7.3.6. □

Corollary 8.4.3. *The equivalence* (8.6) *uniquely extends to an equivalence*

$$\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y}} : \text{D-mod}(\mathcal{Y})^\vee \xrightarrow{\sim} \text{D-mod}(\mathcal{Y}). \tag{8.7}$$

Proof. By Theorem 8.1.1, $\text{D-mod}(\mathcal{Y}) = \mathrm{Ind}(\text{D-mod}(\mathcal{Y})^c)$. By Sect. 4.1.3(ii'), this implies that $\text{D-mod}(\mathcal{Y})^\vee = \mathrm{Ind}((\text{D-mod}(\mathcal{Y})^c)^{\mathrm{op}})$, so

$$\text{D-mod}(\mathcal{Y})^\vee = \mathrm{Ind}((\text{D-mod}(\mathcal{Y})^c)^{\mathrm{op}}) \simeq \mathrm{Ind}(\text{D-mod}(\mathcal{Y})^c) \simeq \text{D-mod}(\mathcal{Y}).$$

□

8.4.4. According to Sect. 4.1.1, the self-duality given by (8.7) corresponds to a pair of functors

$$\epsilon_{\text{D-mod}(\mathcal{Y})} : \text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}) \to \mathrm{Vect} \tag{8.8}$$

and

$$\mu_{\text{D-mod}(\mathcal{Y})} : \mathrm{Vect} \to \text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}). \tag{8.9}$$

We shall also use the notation $\langle -, - \rangle_{\text{D-mod}(\mathcal{Y})}$ to denote the functor

$$\text{D-mod}(\mathcal{Y}) \times \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}) \overset{\epsilon_{\text{D-mod}(\mathcal{Y})}}{\longrightarrow} \mathrm{Vect}\,.$$

From Lemma 7.3.5, we obtain:

Lemma 8.4.5. *For* $\mathcal{M} \in \text{D-mod}(\mathcal{Y})^c$ *and* $\mathcal{M}' \in \text{D-mod}(\mathcal{Y})$ *we have*

$$\langle \mathcal{M}, \mathcal{M}' \rangle_{\text{D-mod}(\mathcal{Y})} = \Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}').$$

In Corollary 9.2.15 we shall describe the functor $\epsilon_{\text{D-mod}(\mathcal{Y})}$ on the entire category

$$\text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}) \simeq \text{D-mod}(\mathcal{Y} \times \mathcal{Y})$$

explicitly. Furthermore, in Sect. 9.2.16 we will prove:

Proposition 8.4.6. *The object*

$$\mu_{\text{D-mod}(\mathcal{Y})}(k) \in \text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}) \simeq \text{D-mod}(\mathcal{Y} \times \mathcal{Y})$$

identifies canonically with $(\Delta_{\mathcal{Y}})_{\mathrm{dR},*}(\omega_{\mathcal{Y}})$.

8.4.7. Let now $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a schematic quasi-compact morphism between QCA stacks. Recall the notion of the dual functor, see Sect. 4.1.4. We claim:

Proposition 8.4.8. *The functors*

$$\pi_{\mathrm{dR},*} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2) \textit{ and } \pi^! : \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1)$$

are related by $(\pi_{\mathrm{dR},*})^\vee \simeq \pi^!$ *in terms of the self-dualities* $\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y}_i} : \text{D-mod}(\mathcal{Y}_i)^\vee \simeq \text{D-mod}(\mathcal{Y}_i)$.

Proof. It suffices to construct a functorial isomorphism for $\mathcal{M}_i \in \text{D-mod}(\mathcal{Y}_i)^c$:

$$\langle \mathcal{M}_2, \pi_{\mathrm{dR},*}(\mathcal{M}_1)\rangle_{\text{D-mod}(\mathcal{Y}_2)} \simeq \langle \pi^!(\mathcal{M}_2), \mathcal{M}_1\rangle_{\text{D-mod}(\mathcal{Y}_1)}.$$

By Lemma 8.4.5 we rewrite the left-hand side as

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_2, \mathcal{M}_2 \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{M}_1)),$$

which by the projection formula (6.2) identifies with

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_2, \pi_{\mathrm{dR},*}(\pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1)).$$

However, by Lemma 7.5.6, the latter identifies with

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_1, \pi^!(\mathcal{M}_2) \overset{!}{\otimes} \mathcal{M}_1),$$

which in turn identifies with $\langle \pi^!(\mathcal{M}_2), \mathcal{M}_1\rangle_{\text{D-mod}(\mathcal{Y}_1)}$ again by Lemma 8.4.5.

□

9. Renormalized de Rham cohomology and safety

As we saw in Sect. 7.1.3, for a QCA algebraic stack $\mathcal{Y}$, the functor $\Gamma_{\mathrm{dR}}(\mathcal{Y}, -)$ is not necessarily continuous. In this section we shall introduce a new functor, denoted $\Gamma_{\text{ren-dR}}(\mathcal{Y}, -)$ that we will refer to as "renormalized de Rham cohomology". This functor will be continuous, and we will have a natural transformation

$$\Gamma_{\text{ren-dR}}(\mathcal{Y}, -) \to \Gamma_{\mathrm{dR}}(\mathcal{Y}, -).$$

We shall also introduce a class of objects on D-mod($\mathcal{Y}$), called *safe*, for which the above natural transformation is an equivalence.

In this section all algebraic stacks will be assumed QCA, unless specified otherwise.

9.1. **Renormalized de Rham cohomology.** Recall the notion of the dual functor from Sect. 4.1.4.

Definition 9.1.1. *For a QCA algebraic stack* $\mathcal{Y}$ *we define the* continuous *functor*

$$\Gamma_{\text{ren-dR}}(\mathcal{Y}, -) : \text{D-mod}(\mathcal{Y}) \to \mathrm{Vect}$$

to be the dual of

$$\pi^!_{\mathcal{Y}} : \mathrm{Vect} \to \text{D-mod}(\mathcal{Y}), \quad k \mapsto \omega_{\mathcal{Y}}$$

under the identifications

$$\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Y}} : \text{D-mod}(\mathcal{Y})^\vee \simeq \text{D-mod}(\mathcal{Y}) \textit{ and } \mathrm{Vect}^\vee \simeq \mathrm{Vect}\,.$$

Note that if $\mathcal{Y}$ is a scheme Z, by (5.17), we have $\Gamma_{\text{ren-dR}}(Z, -) \simeq \Gamma_{\mathrm{dR}}(Z, -)$.

Remark 9.1.2. Presumably, the functor analogous to $\Gamma_{\text{ren-dR}}(Z, -)$ can be defined in other sheaf-theoretic situations, e.g., for the derived category of sheaves with constructible cohomologies for stacks over the field of complex numbers.

9.1.3. Here is a more explicit description of the functor $\Gamma_{\text{ren-dR}}(\mathcal{Y},-)$.

Lemma 9.1.4. *The functor $\Gamma_{\text{ren-dR}}(\mathcal{Y},-)$ (see Sect. 8.4.4 for the notation) is canonically isomorphic to the ind-extension of the functor*

$$\Gamma_{\mathrm{dR}}(\mathcal{Y},-)|_{\text{D-mod}(\mathcal{Y})^c} : \text{D-mod}(\mathcal{Y})^c \to \mathrm{Vect}\,.$$

Proof. We only have to show that the pairing $\langle -,-\rangle_{\text{D-mod}(\mathcal{Y})}$ corresponding to the self-duality of D-mod$(\mathcal{Y})$ satisfies

$$\langle \mathcal{M}, p_{\mathcal{Y}}^!(k)\rangle_{\text{D-mod}(\mathcal{Y})} \simeq \Gamma_{\mathrm{dR}}(\mathcal{Y},\mathcal{M})$$

for $\mathcal{M}\in \text{D-mod}(\mathcal{Y})^c$. However, this is immediate from Lemma 8.4.5. □

Corollary 9.1.5. *There exists a canonically defined natural transformation*

$$\Gamma_{\text{ren-dR}}(\mathcal{Y},-) \to \Gamma_{\mathrm{dR}}(\mathcal{Y},-), \tag{9.1}$$

which is an isomorphism when restricted to compact objects.

In general, the failure of the natural transformation (9.1) to be an isomorphism is a measure to which the functor $\Gamma_{\mathrm{dR}}(\mathcal{Y},-)$ fails to be continuous.

Example 9.1.6. As an illustration, let us compute the functor $\Gamma_{\text{ren-dR}}(\mathcal{Y},-)$ for $\mathcal{Y}=BG$, see Sect. 7.2. Let B be as in (7.4). We saw in *loc.cot.* that the functor $\Gamma_{\mathrm{dR}}(BG,-)$ is given by $\mathcal{M}aps_{B\text{-mod}}(k,-)$.

We claim now that the functor $\Gamma_{\text{ren-dR}}(BG,-)$ is given by

$$M\mapsto k\underset{B}{\otimes} M[-2\dim(G)+\delta],$$

where δ is the degree of the highest cohomology group of $\Gamma_{\mathrm{dR}}(G,k_G)$.

Explicitly,

$$\begin{cases} \delta=0, \text{ if } G \text{ is unipotent;}\\ \delta=\dim(G), \text{ if } G \text{ is reductive;}\\ \delta=2\dim(G), \text{ if } G \text{ is an abelian variety.}\end{cases}$$

Recall that σ denotes the map $\mathrm{pt}\to BG$, and recall that $\sigma_!(k)$ is a compact generator of D-mod(BG). Hence, it suffices to show that

$$\Gamma_{\mathrm{dR}}(BG,\sigma_!(k))\simeq k[-2\dim(G)+\delta],$$

as modules over $B\simeq \mathcal{M}aps_{\text{D-mod}}(\sigma_!(k),\sigma_!(k))$.

Note that

$$\sigma_!(k)\simeq \sigma_{\mathrm{dR},*}(k)[-2\dim(G)+\delta],$$

so the required assertion follows from the isomorphism

$$\Gamma_{\mathrm{dR}}(BG,\sigma_{\mathrm{dR},*}(k))\simeq \Gamma_{\mathrm{dR}}(\mathrm{pt},k)=k.$$

Example 9.1.7. We claim that the functor

$$\Gamma_{\text{ren-dR}}(\mathcal{Y},-)\circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$$

identifies canonically with $\Gamma^{\mathrm{IndCoh}}(\mathcal{Y},-)$.

Both functors are continuous, so it is enough to construct the isomorphism on the subcategory $\mathrm{Coh}(\mathcal{Y})\subset \mathrm{IndCoh}(\mathcal{Y})$. In the latter case the assertion follows from Lemma 9.1.4 and Proposition 7.1.6.

Moreover, we obtain that the natural transformation (9.1) induces an isomorphism

$$\Gamma_{\text{ren-dR}}(\mathcal{Y},-)\circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \to \Gamma_{\text{dR}}(\mathcal{Y},-)\circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}.$$

As we shall see shortly, the latter isomorphism is a general phenomenon that holds for all *safe* objects of $\text{D-mod}(\mathcal{Y})$.

9.2. **Safe objects of** $\text{D-mod}(\mathcal{Y})$**.**

Definition 9.2.1. *An object* $\mathcal{M}\in \text{D-mod}(\mathcal{Y})$ *is said to be* safe *if the functor*

$$\mathcal{M}'\mapsto \Gamma_{\text{dR}}(\mathcal{Y},\mathcal{M}\overset{!}{\otimes}\mathcal{M}') : \text{D-mod}(\mathcal{Y})\to \text{Vect}$$

is continuous.

Its is clear that safe objects of $\text{D-mod}(\mathcal{Y})$ form a (non-cocomplete) DG subcategory (i.e., the condition of being safe survives taking cones).

It is also clear that the subcategory of safe objects in $\text{D-mod}(\mathcal{Y})$ is a tensor ideal with respect to $\overset{!}{\otimes}$. Indeed, if $\mathcal{M}$ is safe, then so are all $\mathcal{M}\overset{!}{\otimes}\mathcal{M}'$.

9.2.2. The notion of safety is what allows us to distinguish compact objects among the larger subcategory $\mathcal{M}\in \text{D-mod}_{\text{coh}}(\mathcal{Y})$:

Proposition 9.2.3. *Then the following properties of an object* $\mathcal{M}\in \text{D-mod}_{\text{coh}}(\mathcal{Y})$ *are equivalent:*

(a) $\mathcal{M}$ *is compact;*
(b) $\mathcal{M}$ *is safe;*
(c) $\mathbb{D}^{\text{Verdier}}_{\mathcal{Y}}(\mathcal{M})$ *is safe.*

Proof. By Lemma 7.3.5, (a) is equivalent to (c). So (b) is equivalent to the compactness of $\mathbb{D}^{\text{Verdier}}_{\mathcal{Y}}(\mathcal{F})$. The latter is equivalent to (a) by Corollary 8.4.2 (it is here that we use that $\mathcal{Y}$ is QCA). □

Note, however, safe objects do not have to be coherent or cohomologically bounded:

Example 9.2.4. We claim that all objects of the form $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})$, $\mathcal{F}\in \text{IndCoh}(\mathcal{Y})$, are safe. Indeed, by Lemma 6.3.20 and Proposition 7.1.6, for $\mathcal{M}\in \text{D-mod}(\mathcal{Y})$

$$\Gamma_{\text{dR}}(\mathcal{Y},\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})\overset{!}{\otimes}\mathcal{M})\simeq \Gamma^{\text{IndCoh}}(\mathcal{Y},\mathcal{F}\overset{!}{\otimes}\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})),$$

and the latter functor is continuous.

9.2.5. The following will be useful in the sequel:

Lemma 9.2.6. *Let* $\pi:\mathcal{Y}_1\to\mathcal{Y}_2$ *be schemtaic. If* $\mathcal{M}_2\in \text{D-mod}(\mathcal{Y}_2)$ *is safe, then so is* $\pi^!(\mathcal{Y}_2)\in \text{D-mod}(\mathcal{Y}_1)$.

Proof. We need to show that the functor

$$\mathcal{M}_1\mapsto \Gamma_{\text{dR}}(\mathcal{Y}_1,\pi^!(\mathcal{M}_2)\overset{!}{\otimes}\mathcal{M}_1)$$

commutes with colimits. By Lemma 7.5.6, the latter expression can be rewritten as

$$\Gamma_{\text{dR}}(\mathcal{Y}_2,\pi_{\text{dR},*}(\pi^!(\mathcal{Y}_2)\overset{!}{\otimes}\mathcal{M}_1)).$$

Now, since π is schematic and quasi-compact, the projection formula and (6.2) is applicable, and we can rewrite the latter expression as

$$\Gamma_{\text{dR}}(\mathcal{Y}_2,\mathcal{M}_2\overset{!}{\otimes}\pi_{\text{dR},*}(\mathcal{M}_1)).$$

Now, the required assertion follows from the fact that the functor $\pi_{\mathrm{dR},*}$ commutes with colimits.

□

Remark 9.2.7. In Lemma 10.4.2 we will extend the assertion of the above lemma to the case when π is not necessarily schematic, but merely *safe*. However, the lemma obviously fails for general morphisms: consider, e.g., $B\mathbb{G}_m \to \mathrm{pt}$.

9.2.8. *De Rham cohomology of safe objects.* The following proposition is crucial for the sequel:

Proposition 9.2.9. *Let $\mathcal{M}_1 \in \mathrm{D\text{-}mod}(\mathcal{Y})$ be safe. Then for any $\mathcal{M}_2 \in \mathrm{D\text{-}mod}(\mathcal{Y})$, the natural transformation* (9.1) *induces an isomorphism*

$$\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2) \to \Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2).$$

Proof. By Lemma 9.1.4, we have:

$$\tau^{\leq 0}\left(\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2)\right) \simeq \underset{\mathcal{M} \in \mathrm{D\text{-}mod}(\mathcal{Y})^c_{/\mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2}}{\underrightarrow{colim}} \tau^{\leq 0}\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M})\right). \tag{9.2}$$

Using the fact that

$$\tau^{\leq 0}\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M})\right) \simeq \mathrm{Maps}_{\mathrm{D\text{-}mod}(\mathcal{Y})}(k_{\mathcal{Y}}, \mathcal{M}),$$

we can rewrite (9.2) as the co-end of the functors

$$\mathcal{M} \mapsto \mathrm{Maps}_{\mathrm{D\text{-}mod}(\mathcal{Y})}(\mathcal{M}, \mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2) \text{ and } \mathcal{M} \mapsto \mathrm{Maps}_{\mathrm{D\text{-}mod}(\mathcal{Y})}(k_{\mathcal{Y}}, \mathcal{M})$$

out of $\mathrm{D\text{-}mod}(\mathcal{Y})^c$. Using the Verdier duality anti-equivalence of $\mathrm{D\text{-}mod}(\mathcal{Y})^c$, we rewrite the above co-end as the co-end of the functors

$$\mathcal{M}' \mapsto \tau^{\leq 0}\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2 \overset{!}{\otimes} \mathcal{M}')\right) \text{ and } \mathcal{M}' \mapsto \mathrm{Maps}_{\mathrm{D\text{-}mod}(\mathcal{Y})}(\mathcal{M}', \omega_{\mathcal{Y}}),$$

as functors out of $(\mathrm{D\text{-}mod}(\mathcal{Y})^c)^{\mathrm{op}}$.

The latter co-end can be rewritten as

$$\underset{\mathcal{M}' \in \mathrm{D\text{-}mod}(\mathcal{Y})^c_{/\omega_{\mathcal{Y}}}}{\underrightarrow{colim}} \tau^{\leq 0}(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2 \overset{!}{\otimes} \mathcal{M}')). \tag{9.3}$$

However, tautologically,

$$\underset{\mathcal{M}' \in \mathrm{D\text{-}mod}(\mathcal{Y})^c_{/\omega_{\mathcal{Y}}}}{\underrightarrow{colim}} \mathcal{M}' \simeq \omega_{\mathcal{Y}},$$

and hence

$$\underset{\mathcal{M}' \in \mathrm{D\text{-}mod}(\mathcal{Y})^c_{/\omega_{\mathcal{Y}}}}{\underrightarrow{colim}} \mathcal{M}_2 \overset{!}{\otimes} \mathcal{M}' \simeq \mathcal{M}_2.$$

Hence, the assumption that $\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M}_1 \overset{!}{\otimes} -)$ commutes with colimits implies that the expression in (9.3) maps isomorphically to $\tau^{\leq 0}\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M}_1 \overset{!}{\otimes} \mathcal{M}_2)\right)$, as required.

□

As a particular case, we obtain:

Corollary 9.2.10. *If* $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$ *is safe, the natural transformation* (9.1) *induces an isomorphism*

$$\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M}) \to \Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M}).$$

In addition:

Corollary 9.2.11. *An object* $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$ *is safe if and only if the natural transformation* (9.1) *induces an isomorphism*

$$\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}') \to \Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}')$$

for any $\mathcal{M}' \in \text{D-mod}(\mathcal{Y})$.

Combining Proposition 9.2.9 with Proposition 9.2.3, we obtain:

Corollary 9.2.12. *If one of the objects* $\mathcal{M}'$ *or* $\mathcal{M}''$ *is compact, then the map*

$$\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M}' \overset{!}{\otimes} \mathcal{M}'') \to \Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M}' \overset{!}{\otimes} \mathcal{M}'')$$

is an isomorphism.

9.2.13. The notion of safe object allows to give a more explicit description of the pairing $\langle -, - \rangle_{\text{D-mod}(\mathcal{Y})}$:

Lemma 9.2.14. *For* $\mathcal{M}', \mathcal{M}'' \in \text{D-mod}(\mathcal{Y})$*, the natural map*

$$\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M}' \overset{!}{\otimes} \mathcal{M}'') \to \langle \mathcal{M}', \mathcal{M}'' \rangle_{\text{D-mod}(\mathcal{Y})}$$

is an isomorphism.

Proof. By definition, both functors in the corollary are continuous, so it is enough to verify the assertion for $\mathcal{M}'$ and $\mathcal{M}''$ compact. By definition, the right-hand side is $\Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M}' \overset{!}{\otimes} \mathcal{M}'')$. So, the assertion follows from Corollary 9.2.12. $\square$

Corollary 9.2.15. *The functor*

$$\epsilon_{\text{D-mod}(\mathcal{Y})} : \text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}) \to \text{Vect}$$

identifes canonically with

$$\text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}) \simeq \text{D-mod}(\mathcal{Y} \times \mathcal{Y}) \xrightarrow{\Delta^!_{\mathcal{Y}}} \text{D-mod}(\mathcal{Y}) \xrightarrow{\Gamma_{\text{ren-dR}}(\mathcal{Y}, -)} \text{Vect}\,.$$

9.2.16. *Proof of Proposition 8.4.6.* The functor

$$\mu_{\text{D-mod}(\mathcal{Y})} : \text{Vect} \to \text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y}) \simeq \text{D-mod}(\mathcal{Y} \times \mathcal{Y})$$

is the dual of the functor $\epsilon_{\text{D-mod}(\mathcal{Y})}$ under the identifications

$$\text{Vect}^\vee \simeq \text{Vect} \ \text{ and } \mathbf{D}^{\text{Verdier}}_{\mathcal{Y} \times \mathcal{Y}} : \text{D-mod}(\mathcal{Y} \times \mathcal{Y})^\vee \simeq \text{D-mod}(\mathcal{Y} \times \mathcal{Y}).$$

Hence, the required assertion follows from Corollary 9.2.15 and Proposition 8.4.8.

$\square$

9.3. **The relative situation.**

9.3.1. Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between QCA algebraic stacks, and consider the functor $\pi^! : \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1)$.

Definition 9.3.2. *We define the* continuous *functor*

$$\pi_\blacktriangle : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2)$$

to be the dual of $\pi^!$ *under the identifications* $\text{D-mod}(\mathcal{Y}_i)^\vee \simeq \text{D-mod}(\mathcal{Y}_i)$.

We shall refer to the functor $\pi_\blacktriangle$ as the "renormalized direct image".

Note that Proposition 8.4.8 implies that if π is schematic, then $\pi_\blacktriangle \simeq \pi_{\text{dR},*}$.

9.3.3. It follows from the construction that the assignment $\pi \rightsquigarrow \pi_\blacktriangle$ is compatible with compositions, i.e., for

$$\mathcal{Y}_1 \xrightarrow{\pi} \mathcal{Y}_2 \xrightarrow{\phi} \mathcal{Y}_3$$

there exists a canonical isomorphism

$$\phi_\blacktriangle \circ \pi_\blacktriangle \simeq (\phi \circ \pi)_\blacktriangle.$$

Indeed, this isomorphism follows by duality from $\pi^! \circ \phi^! \simeq (\phi \circ \pi)^!$.

9.3.4. We claim that the functor $\pi_\blacktriangle$ satisfies the projection formula *by definition*:

Lemma 9.3.5. *For* $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$ *and* $\mathcal{M}_2 \in \text{D-mod}(\mathcal{Y}_2)$ *we have a canonical isomorphism*

$$\pi_\blacktriangle(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2 \simeq \pi_\blacktriangle(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)),$$

functorial in $\mathcal{M}_i \in \text{D-mod}(\mathcal{Y}_i)$.

Proof. It suffices to construct a functorial isomorphism

$$\langle \pi_\blacktriangle(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2, \mathcal{M}'_2 \rangle_{\text{D-mod}(\mathcal{Y}_2)} \simeq \langle \pi_\blacktriangle(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)), \mathcal{M}'_2 \rangle_{\text{D-mod}(\mathcal{Y}_2)}$$

functorial in $\mathcal{M}_2, \mathcal{M}'_2 \in \text{D-mod}(\mathcal{Y}_2)$, $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$.

By Lemma 9.2.14,

$$\langle \pi_\blacktriangle(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2, \mathcal{M}'_2 \rangle_{\text{D-mod}(\mathcal{Y}_2)} \simeq \Gamma_{\text{ren-dR}}(\mathcal{Y}_2, \pi_\blacktriangle(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2 \overset{!}{\otimes} \mathcal{M}'_2) \simeq$$
$$\simeq \langle \pi_\blacktriangle(\mathcal{M}_1), \mathcal{M}_2 \overset{!}{\otimes} \mathcal{M}'_2 \rangle_{\text{D-mod}(\mathcal{Y}_2)}.$$

By the definition of $\pi_\blacktriangle$, the latter identifies with

$$\langle \mathcal{M}_1, \pi^!(\mathcal{M}_2 \overset{!}{\otimes} \mathcal{M}'_2) \rangle_{\text{D-mod}(\mathcal{Y}_1)}.$$

Again, by Lemma 9.2.14, the latter expression can be rewritten as

$$\Gamma_{\text{ren-dR}}(\mathcal{Y}_1, \mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2 \overset{!}{\otimes} \mathcal{M}'_2)) \simeq \langle \mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2), \pi^!(\mathcal{M}'_2) \rangle_{\text{D-mod}(\mathcal{Y}_1)},$$

and again by the definition of $\pi_\blacktriangle$, further as

$$\langle \pi_\blacktriangle(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)), \mathcal{M}'_2 \rangle_{\text{D-mod}(\mathcal{Y}_2)},$$

as required.

$\square$

9.3.6. *Calculating* $\pi_\blacktriangle$. It turns out that safe objects are adjusted to calculating the functor $\pi_\blacktriangle$:

Proposition 9.3.7. *There is a canonical natural transformation*

$$\pi_\blacktriangle \to \pi_{\mathrm{dR},*}, \tag{9.4}$$

which is an isomorphism when evaluated on safe objects.

Proof. We need to show that for $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$ and $\mathcal{M}_2 \in \text{D-mod}(\mathcal{Y}_2)^c$ there exists a canonical map

$$\langle \mathcal{M}_1, \pi^!(\mathcal{M}_2)\rangle_{\text{D-mod}(\mathcal{Y}_1)} \to \langle \pi_{\mathrm{dR},*}(\mathcal{M}_1), \mathcal{M}_2\rangle_{\text{D-mod}(\mathcal{Y}_2)}, \tag{9.5}$$

which is an isomorphism if $\mathcal{M}_1$ is safe.

By Lemma 9.2.14 the left-hand side in (9.5) identifies with

$$\Gamma_{\text{ren-dR}}(\mathcal{Y}_1, \mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)).$$

The latter expression maps to

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_1, \mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)),$$

and by Proposition 9.2.9, this map is an isomorphism if $\mathcal{M}_1$ is safe.

By Lemma 9.2.14 and Corollary 9.2.12, the right-hand side in (9.5) identifies with

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_2, \pi_{\mathrm{dR},*}(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2).$$

Thus, we obtain the following diagram of maps

$$\begin{aligned} &\langle \mathcal{M}_1, \pi^!(\mathcal{M}_2)\rangle_{\text{D-mod}(\mathcal{Y}_1)} \simeq \Gamma_{\text{ren-dR}}(\mathcal{Y}_1, \mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)) \to \\ &\to \Gamma_{\mathrm{dR}}(\mathcal{Y}_1, \mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)) \simeq \Gamma_{\mathrm{dR}}(\mathcal{Y}_2, \pi_{\mathrm{dR},*}(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2))) \leftarrow \\ &\leftarrow \Gamma_{\mathrm{dR}}(\mathcal{Y}_2, \pi_{\mathrm{dR},*}(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2) \simeq \langle \pi_{\mathrm{dR},*}(\mathcal{M}_1), \mathcal{M}_2\rangle_{\text{D-mod}(\mathcal{Y}_2)}, \end{aligned}$$

where the left-pointing arrow comes from (7.17). The assertion of the proposition follows now from Lemma 7.8.2.

□

Corollary 9.3.8. *The functor* $\pi_\blacktriangle$ *is canonically isomorphic to the ind-extension of the functor*

$$\pi_{\mathrm{dR},*}|_{\text{D-mod}(\mathcal{Y}_1)^c} : \text{D-mod}(\mathcal{Y}_1)^c \to \text{D-mod}(\mathcal{Y}_2).$$

9.3.9. As another corollary of Proposition 9.3.7, we obtain:

Corollary 9.3.10. *For* $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$ *and* $\mathcal{M}_2 \in \text{D-mod}(\mathcal{Y}_2)$, *the map*

$$\pi_{\mathrm{dR},*}(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2 \to \pi_{\mathrm{dR},*}(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2))$$

of (7.17) *is an isomorphism provided that* $\mathcal{M}_1$ *is safe.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} \pi_\blacktriangle(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2 & \longrightarrow & \pi_\blacktriangle(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)) \\ \downarrow & & \downarrow \\ \pi_{\mathrm{dR},*}(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{M}_2 & \longrightarrow & \pi_{\mathrm{dR},*}(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{M}_2)), \end{array}$$

in which the upper horizontal arrow is an isomorphism by Lemma 9.3.5. Now, the vertical arrows are isomorphisms by Proposition 9.3.7. □

9.3.11. *Base change for the renormalized direct image.* Consider a Cartesian diagram of QCA algebraic stacks:

$$\begin{tikzcd} \mathcal{Y}'_1 \arrow[r, "\phi_1"] \arrow[d, "\pi'"'] & \mathcal{Y}_1 \arrow[d, "\pi"] \\ \mathcal{Y}'_2 \arrow[r, "\phi_2"] & \mathcal{Y}_2 \end{tikzcd}$$

We claim that there exists a canonical natural transformation

$$\pi'_{\blacktriangle} \circ \phi_1^! \to \phi_2^! \circ \pi_{\blacktriangle}. \tag{9.6}$$

Indeed, both functors being continuous, it is enough to construct the morphism in question on $\text{D-mod}(\mathcal{Y}_1)^c$. By (7.16) for any $\mathcal{M} \in \text{D-mod}(\mathcal{Y}_1)$ we have a map

$$\phi_2^! \circ \pi_{\mathrm{dR},*}(\mathcal{M}) \to \pi'_{\mathrm{dR},*} \circ \phi_1^!(\mathcal{M}).$$

Moreover, by Proposition 7.6.8, this map is an isomorphism whenever $\mathcal{M} \in \text{D-mod}(\mathcal{Y}_1)^+$.

Thus, for $\mathcal{M} \in \text{D-mod}(\mathcal{Y}_1)^c$ we have

$$\phi_2^! \circ \pi_{\blacktriangle}(\mathcal{M}) \simeq \phi_2^! \circ \pi_{\mathrm{dR},*}(\mathcal{M}) \simeq \pi'_{\mathrm{dR},*} \circ \phi_1^!(\mathcal{M}),$$

and the latter receives a map from $\pi'_{\blacktriangle} \circ \phi_1^!(\mathcal{M})$.

We now claim:

Proposition 9.3.12. *The map* (9.6) *is an isomorphism.*

Proof. By transitivity and the definition of $\text{D-mod}(\mathcal{Y}'_2)$, it is enough to show that the map in question is an isomorphism when $\mathcal{Y}'_2$ is a DG scheme. In particular, in this case the morphism ϕ_2, and hence ϕ_1, is schematic. However, in this case for $\mathcal{M} \in \text{D-mod}(\mathcal{Y}_1)^c$, the object

$$\phi_1^!(\mathcal{M}) \in \text{D-mod}(\mathcal{Y}'_1)$$

is safe, by Lemma 9.2.6. Therefore, the map

$$\pi'_{\blacktriangle} \circ \phi_1^!(\mathcal{M}) \to \pi'_{\mathrm{dR},*} \circ \phi_1^!(\mathcal{M}),$$

used in the construction of (9.6), is an isomorphism, by Proposition 9.3.7. ∎

Remark 9.3.13. Using (9.6) one can extend the functor $\pi_{\blacktriangle}$ to arbitrary QCA morphisms $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ between prestacks in a way compatible with base change.

In the course of the proof of Proposition 9.3.12 we have also established:

Corollary 9.3.14. *If* $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$ *is safe, then the map* $\phi_2^! \circ \pi_{\mathrm{dR},*}(\mathcal{M}_1) \to \pi'_{\mathrm{dR},*} \circ \phi_1^!(\mathcal{M}_1)$ *of* (7.16) *is an isomorphism.*

9.3.15. *Renormalized direct image of induced D-modules.* Generalizing Example 9.1.7, we claim:

Proposition 9.3.16. *There exists a canonical isomorphism of functors*

$$\pi_{\blacktriangle} \circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_1)} \simeq \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_2)} \circ \pi_*^{\mathrm{IndCoh}}.$$

Proof. Since both functors are continuous, it suffices to show that for $\mathcal{F}_1 \in \mathrm{IndCoh}(\mathcal{Y}_1)^c$ and $\mathcal{M}_2 \in \text{D-mod}(\mathcal{Y}_2)^c$, there exists a canonical isomorphism

$$\langle \pi_{\blacktriangle}(\mathbf{ind}_{\text{D-mod}(\mathcal{Y}_1)}(\mathcal{F}_1)), \mathcal{M}_2 \rangle_{\text{D-mod}(\mathcal{Y}_2)} \simeq \langle \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_2)}(\pi_*^{\mathrm{IndCoh}}(\mathcal{F}_1)), \mathcal{M}_2 \rangle_{\text{D-mod}(\mathcal{Y}_2)}.$$

By the definition of $\pi_{\blacktriangle}$, Lemma 9.2.14 and Corollary 9.2.12, the left-hand side identifies with

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_1, \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_1)}(\mathcal{F}_1) \overset{!}{\otimes} \pi^!(\mathcal{M}_2)),$$

while the right-hand side identifies with

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_2, \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_2)}(\pi_*^{\mathrm{IndCoh}}(\mathcal{F}_1)) \overset{!}{\otimes} \mathcal{M}_2).$$

Using Lemma 6.3.20, the two expressions can be rewritten as

$$\Gamma_{\mathrm{dR}}\left(\mathcal{Y}_1, \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_1)}(\mathcal{F}_1 \overset{!}{\otimes} \pi^!(\mathbf{oblv}_{\text{D-mod}(\mathcal{Y}_2)}(\mathcal{M}_2)))\right)$$

and

$$\Gamma_{\mathrm{dR}}\left(\mathcal{Y}_2, \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_2)}(\pi_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \overset{!}{\otimes} \mathbf{oblv}_{\text{D-mod}(\mathcal{Y}_1)}(\mathcal{M}_2))\right),$$

respectively, and further, using Proposition 7.1.6 as

$$\Gamma^{\mathrm{IndCoh}}\left(\mathcal{Y}_1, \mathcal{F}_1 \overset{!}{\otimes} \pi^!(\mathbf{oblv}_{\text{D-mod}(\mathcal{Y}_2)}(\mathcal{M}_2))\right)$$

and

$$\Gamma^{\mathrm{IndCoh}}\left(\mathcal{Y}_2, \pi^{\mathrm{IndCoh}_*}(\mathcal{F}_1) \overset{!}{\otimes} \mathbf{oblv}_{\text{D-mod}(\mathcal{Y}_1)}(\mathcal{M}_2)\right),$$

respectively.

Now, the required isomorphism follows from Proposition 4.4.11.

□

9.3.17. We are now ready to prove Proposition 7.5.9 stated in Sect. 7.4. Indeed, it follows by combining Propositions 9.3.16, 9.3.7, 3.7.11 and Example 9.2.4.

□

9.4. **Cohomological amplitudes.**

9.4.1. Let us note that by Lemma 7.6.8, the functor $\pi_{\mathrm{dR},*}$ is *left t-exact up to a cohomological shift.* We claim that the functor $\pi_\blacktriangle$ exhibits an opposite behavior:

Proposition 9.4.2. *There exists an integer m such that $\pi_\blacktriangle$ sends*

$$\text{D-mod}(\mathcal{Y}_1)^{\leq 0} \to \text{D-mod}(\mathcal{Y}_2)^{\leq m}.$$

Proof. It is clear from the $(\mathbf{ind}_{\text{D-mod}}, \mathbf{oblv}_{\text{D-mod}})$ adjunction that for any algebraic stack $\mathcal{Y}$, the category $\text{D-mod}(\mathcal{Y})^{\leq 0}$ is generated under the operation of taking filtered colimits by objects of the form $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})$ for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y})^{\leq 0}$.

Hence, by Proposition 9.3.16, it suffices to show that there exists an integer m, such that π_*^{IndCoh} sends

$$\mathrm{IndCoh}(\mathcal{Y}_1)^{\leq 0} \to \mathrm{IndCoh}(\mathcal{Y}_2)^{\leq m}.$$

Recall that the functor $\Psi_{\mathcal{Y}}$ induces an equivalence $\mathrm{IndCoh}(\mathcal{Y})^+ \to \mathrm{QCoh}(\mathcal{Y})^+$ for any algebraic stack $\mathcal{Y}$. Therefore, it suffices to show that there exists an integer m, such that π_* sends

$$\mathrm{QCoh}(\mathcal{Y}_1)^{\leq 0} \to \mathrm{QCoh}(\mathcal{Y}_2)^{\leq m}.$$

However, this follows from Corollary 1.4.5(ii).

□

9.4.3. Let us observe that the safety of an object $\mathcal{M} \in \text{D-mod}(\mathcal{Y})^b$ makes both functors

$$\Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} -) \text{ and } \Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} -)$$

cohomologically bounded. More precisely:

Lemma 9.4.4.
(a) *Let $\mathcal{M}$ be an safe object of* $\text{D-mod}(\mathcal{Y})^-$*. Then the functor*

$$\mathcal{M}_1 \mapsto \Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)$$

is right t-exact up to a cohomological shift. The estimate on the shift depends only on $\mathcal{Y}$ and the integer m such that $\mathcal{M} \in \text{D-mod}(\mathcal{Y})^{\leq m}$.
(b) *Let $\mathcal{M}$ be a safe object of* $\text{D-mod}(\mathcal{Y})^+$*. Then the functor*

$$\mathcal{M}_1 \mapsto \Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)$$

is left t-exact up to a cohomological shift. The estimate on the shift depends only on $\mathcal{Y}$ and the integer m such that $\mathcal{M} \in \text{D-mod}(\mathcal{Y})^{\geq -m}$.

Proof. It is easy to see that on *any* quasi-compact algebraic stack, the functor $\overset{!}{\otimes}$ is both left and right t-exact up to a cohomological shift. Both assertions follow from the fact that if $\mathcal{M}$ is safe,

$$\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1) \simeq \Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)$$

(by Corollary 9.2.11), using Lemma 7.6.8 and Proposition 9.4.2, respectively. $\square$

We now claim that the above lemma admits a partial converse:

Proposition 9.4.5.
(a) *Let $\mathcal{M}$ be an object of* $\text{D-mod}(\mathcal{Y})^b$*. Then it is safe if the functor*

$$\mathcal{M}_1 \mapsto \Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)$$

is right t-exact up to a cohomological shift.
(b) *Let $\mathcal{M}$ be an object of* $\text{D-mod}(\mathcal{Y})^b$*. Then it is safe if the functor*

$$\mathcal{M}_1 \mapsto \Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)$$

is left t-exact up to a cohomological shift.

Combing this with Lemma 9.4.4, we obtain:

Corollary 9.4.6. *Let $\mathcal{M}_i$ be a (possibly infinite) collection of safe objects of* $\text{D-mod}(\mathcal{Y})$ *that are contained in* $\text{D-mod}(\mathcal{Y})^{\geq -m, \leq m}$ *for some m. Then $\underset{i}{\oplus} \mathcal{M}_i$ is also safe.*

Proof of Proposition 9.4.5. To prove point (a), it suffices to show that the functor

$$\mathcal{M}_1 \mapsto H^0\left(\Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)\right)$$

commutes with direct sums. Let k be the integer such that the functor

$$\mathcal{M}_1 \mapsto \Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)$$

sends $\text{D-mod}(\mathcal{Y})^{\leq 0} \to \text{Vect}^{\leq k}$. Let d be an integer such that $\overset{!}{\otimes}$ sends

$$\text{D-mod}(\mathcal{Y})^{\leq 0} \times \text{D-mod}(\mathcal{Y})^{\leq 0} \to \text{D-mod}(\mathcal{Y})^{\leq d}.$$

For a family of objects $\alpha \mapsto \mathcal{M}_1^\alpha$, consider the following diagram in which the columns are parts of long exact sequences:

$$\begin{CD}
\underset{\alpha}{\oplus} H^0\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \tau^{<-k-d}(\mathcal{M}_1^\alpha))\right) @>>> H^0\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} (\underset{\alpha}{\oplus} \tau^{<-k-d}(\mathcal{M}_1^\alpha)))\right) \\
@VVV @VVV \\
\underset{\alpha}{\oplus} H^0\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1^\alpha)\right) @>>> H^0\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} (\underset{\alpha}{\oplus} \mathcal{M}_1^\alpha))\right) \\
@VVV @VVV \\
\underset{\alpha}{\oplus} H^0\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \tau^{\geq -k-d}(\mathcal{M}_1^\alpha))\right) @>>> H^0\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} (\underset{\alpha}{\oplus} \tau^{\geq -k-d}(\mathcal{M}_1^\alpha)))\right) \\
@VVV @VVV \\
\underset{\alpha}{\oplus} H^1\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \tau^{<-k-d}(\mathcal{M}_1^\alpha))\right) @>>> H^1\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} (\underset{\alpha}{\oplus} \tau^{<-k-d}(\mathcal{M}_1^\alpha)))\right).
\end{CD}$$

The top and the bottom horizontal arrows are maps between zero objects by assumption. Hence, the middle vertical arrows in both columns are isomorphisms. The second from the bottom horizontal arrow is an isomorphism by Lemma 7.6.8. Hence, the second from the top horizontal arrow is also an isomorphism, as required.

Let us now prove point (b). We shall show that under the assumptions on $\mathcal{M}$, the functor $\mathcal{M}_1 \mapsto \Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)$ is *right* t-exact, up to a cohomological shift, thereby reducing the assertion to point (a).

Let n be the integer such that

$$H^i\left(\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)\right) = 0$$

for all $i > n$ and $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y})^{\leq 0}$. Such an integer exists because $\mathcal{M}$ is bounded and the functor $\Gamma_{\text{ren-dR}}(\mathcal{Y}, -)$ is *right t-exact* up to a cohomological shift.

We will show that the same integer works for $\Gamma_{\mathrm{dR}}(\mathcal{Y}, -)$, i.e.,

$$H^i\left(\Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)\right) = 0$$

for all $i > n$ and $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y})^{\leq 0}$.

First, we claim that it is sufficient to show this for $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y})^{\heartsuit}$. Indeed, it is clear that the assertion for $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y})^{\heartsuit}$ implies the assertion for all $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y})^b \cap \text{D-mod}(\mathcal{Y})^{\leq 0}$. In general, we use the fact that the functor $\Gamma_{\mathrm{dR}}(\mathcal{Y}, -)$ commutes with limits and the fact that for $\mathcal{M} \in \text{D-mod}(\mathcal{Y})^b$ and any $\mathcal{M}_1$, the map

$$\mathcal{M} \overset{!}{\otimes} \mathcal{M}_1 \to \underset{\underset{m}{\longleftarrow}}{lim} \, (\mathcal{M} \overset{!}{\otimes} \tau^{\geq -m}(\mathcal{M}_1))$$

is an isomorphism (which in turn follows from the fact that the t-structure on $\text{D-mod}(\mathcal{Y})$ is left-complete, and the functor $\mathcal{M} \overset{!}{\otimes} -$ is of bounded cohomological amplitude).

Next, by Lemma 7.6.8(b), we can assume that $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y})^\heartsuit \cap \text{D-mod}_{\text{coh}}(\mathcal{Y})$. We will show that

$$H^i\left(\Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)\right) = 0$$

for all $i > n$ and $\mathcal{M}_1 \in \text{D-mod}_{\text{coh}}(\mathcal{Y})^{\leq 0}$.

We shall use the following lemma, proved in Sect. 9.4.8:

Lemma 9.4.7. *Let $\mathcal{Y}$ be a QCA stack, and $\mathcal{N}$ an object of* $\text{D-mod}_{\text{coh}}(\mathcal{Y})$.
(a) *For a given integer k there exists $\mathcal{N}^k \in$* $\text{D-mod}(\mathcal{Y})^c$ *equipped with a map $\mathcal{N}^k \to \mathcal{N}$, whose cone belongs to* $\text{D-mod}(\mathcal{Y})^{\leq -k}$.
(b) *For a given integer k there exists $\mathcal{N}^k \in$* $\text{D-mod}(\mathcal{Y})^c$ *equipped with a map $\mathcal{N} \to \mathcal{N}^k$, whose cone belongs to* $\text{D-mod}(\mathcal{Y})^{\geq k}$.

For $\mathcal{M}_1 \in \text{D-mod}_{\text{coh}}(\mathcal{Y})^{\leq 0}$, let $\mathcal{M}_1 \to \mathcal{M}_1^k$ be as in Lemma 9.4.7(b). Let

$$\mathcal{L}^k := \text{Cone}(\mathcal{M}_1 \to \mathcal{M}_1^k).$$

Consider the diagram

$$\begin{tikzcd}
\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1) \arrow[r] \arrow[d] & \Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1) \arrow[d] \\
\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1^k) \arrow[r] \arrow[d] & \Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1^k) \arrow[d] \\
\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{L}^k) \arrow[r] & \Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{L}^k),
\end{tikzcd}$$

in which the columns are exact triangles. The middle horizontal arrow is an isomorphism by Corollary 9.2.12. We now claim that for $j = i$ and $i-1$ (or any finite range of indices) and $k \gg 0$, both

$$H^j\left(\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{L}^k)\right) \text{ and } H^j\left(\Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{L}^k)\right)$$

are zero. Indeed, for $\Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{L}^k)$ this follows from Lemma 7.6.8. For $\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{L}^k)$ this follows on the assumption on $\mathcal{M}$.

Hence, $H^i(\Gamma_{\text{ren-dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1)) \to H^i(\Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \mathcal{M}_1))$ is an isomorphism, and the assertion follows.

□

9.4.8. *Proof of Lemma 9.4.7.* To prove point (a), the usual argument reduces the assertion to the following one:

For $\mathcal{N} \in \text{D-mod}(\mathcal{Y})^\heartsuit \cap \text{D-mod}_{\text{coh}}(\mathcal{Y})$, there exists an object $\mathcal{N}_0 \in \text{D-mod}(\mathcal{Y})^{\leq 0} \cap \text{D-mod}(\mathcal{Y})^c$ and a *surjective* map in $\text{D-mod}(\mathcal{Y})^\heartsuit$:

$$H^0(\mathcal{N}_0) \to \mathcal{N}.$$

Write

$$H^0\left(\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{N})\right) = \underset{\alpha}{\cup} \mathcal{F}_\alpha, \quad \mathcal{F}_\alpha \in \text{Coh}(\mathcal{Y})^\heartsuit.$$

The objects $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}_\alpha)$ are compact, and for some α the resulting map

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}_\alpha) \to \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}(\mathcal{N})) \to \mathcal{N}$$

will induce a surjection on H^0.

Point (b) is obtained from point (a) by Verdier duality, using the fact that

$$\mathbb{D}^{\text{Verdier}}_{\mathcal{Y}}\left(\text{D-mod}_{\text{coh}}(\mathcal{Y})^{\leq 0}\right) \subset \text{D-mod}_{\text{coh}}(\mathcal{Y})^{\geq -d}$$

for some integer d depending only on $\mathcal{Y}$.

□

9.5. **Expressing $(\text{dR}, *)$-pushforward through the renormalized version.**

9.5.1. Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be again a morphism between QCA stacks. One can regard the functor $\pi_\blacktriangle$ as a fundamental operation, and wonder whether one can recover the functor $\pi_{\text{dR},*}$ intrinsically through it.

The latter turns out to be possible, once we take into account the t-structure on $\text{D-mod}(\mathcal{Y}_i)$, and below we explain how to do it.

9.5.2. First, according to Lemma 7.5.12, the functor $\pi_{\text{dR},*}$ can be recovered from its restriction to $\text{D-mod}(\mathcal{Y}_1)^{\geq -n}$ for every fixed n, by taking the limit of its values on the truncations.

Second, according to Proposition 7.6.8(b), the restriction of $\pi_{\text{dR},*}$ to $\text{D-mod}(\mathcal{Y}_1)^{\geq -n}$ commutes with filtered colimits, while $\text{D-mod}(\mathcal{Y}_1)^{\geq -n}$ is generated under filtered colimits by the subcategory $\text{D-mod}(\mathcal{Y}_1)^{\geq -n} \cap \text{D-mod}_{\text{coh}}(\mathcal{Y}_1)$.

Hence, it remains to show how to express $\pi_{\text{dR},*}|_{\text{D-mod}_{\text{coh}}(\mathcal{Y}_1)}$ in terms of $\pi_\blacktriangle|_{\text{D-mod}_{\text{coh}}(\mathcal{Y}_1)}$.

9.5.3. Let $\mathcal{N}$ be an object of $\text{D-mod}_{\text{coh}}(\mathcal{Y}_1)$. For an integer $k \gg 0$, let $\mathcal{N} \to \mathcal{N}^k$ be as in Lemma 9.4.7(b). Note that since $\mathcal{N}^k$ is compact, the map

$$\pi_\blacktriangle(\mathcal{N}^k) \to \pi_{\text{dR},*}(\mathcal{N}^k)$$

is an isomorphism.

From Proposition 7.6.8(b) we obtain:

Lemma 9.5.4. *There exists an integer m, depending only on π, such that the map $\mathcal{N} \to \mathcal{N}^k$ induces an isomorphism*

$$\tau^{\leq k-m}(\pi_{\text{dR},*}(\mathcal{N})) \to \tau^{\leq k-m}(\pi_{\text{dR},*}(\mathcal{N}^k)).$$

9.5.5. The above procedure can be summarized as follows:

Proposition 9.5.6.

(a) *The functor $\pi_{\text{dR},*}$ maps isomorphically to the right Kan extension of its retsriction to* $\text{D-mod}(\mathcal{Y}_1)^+$.

(b) *For every n, the restriction of $\pi_{\text{dR},*}$ to* $\text{D-mod}(\mathcal{Y}_1)^{\geq -n}$ *receives an isomorphism from the left Kan extension of its further restriction to* $\text{D-mod}(\mathcal{Y}_1)^{\geq -n} \cap \text{D-mod}_{\text{coh}}(\mathcal{Y}_1)$.

(c) *The restriction of $\pi_{\text{dR},*}$ to* $\text{D-mod}_{\text{coh}}(\mathcal{Y}_1)$ *maps isomorphically to the right Kan extension of its further restriction to* $\text{D-mod}(\mathcal{Y}_1)^c$.

Recall that the restrictions of $\pi_{\text{dR},*}$ and $\pi_\blacktriangle$ to $\text{D-mod}(\mathcal{Y}_1)^c$ are canonically equivalent. So the above proposition indeed expresses $\pi_{\text{dR},*}$ in terms of $\pi_\blacktriangle$.

10. Geometric criteria for safety

10.1. **Overview of the results.** The results of this section have to do with a more explicit description of the subcategory of safe objects in $\text{D-mod}(\mathcal{Y})$. By Proposition 9.2.3, this decsription will characterize the subcategory $\text{D-mod}(\mathcal{Y})^c$ inside $\text{D-mod}_{\text{coh}}(\mathcal{Y})$.

We will introduce a notion of *safe* algebraic stack (see Definition 10.2.2). We will show that a quasi-compact algebraic stack $\mathcal{Y}$ is safe if and only if all objects of $\text{D-mod}(\mathcal{Y})$ are safe. In particular, for a quasi-compat $\mathcal{Y}$, the equality $\text{D-mod}(\mathcal{Y})^c = \text{D-mod}_{\text{coh}}(\mathcal{Y})$ holds if and only if $\mathcal{Y}$ is safe.

For an arbitrary QCA stack $\mathcal{Y}$ we shall formulate an explicit safety criterion for objects of $\text{D-mod}(\mathcal{Y})$ (Theorem 10.2.9). Note that safety for objects can be checked strata-wise (see Corollary 10.4.3).

This section is organized as follows. In Sect. 10.2 we formulate the results and give some easy proofs. The more difficult Theorems 10.2.4 and 10.2.9 are proved in Sect. 10.3-10.5.

As we shall be only interested in the categorical aspects of $\text{D-mod}(\mathcal{Y})$, with no restriction of generality we can assume that all schemes and algebraic stacks discussed in this section are classical.

Change of conventions: For the duration of this section "prestack" will mean "classical prestack", and "algebraic stack" will mean "classical algebraic stack".

10.2. **Formulations.**

10.2.1. *Safe algebraic stacks and morphisms.*

Definition 10.2.2.

(a) *An algebraic stack $\mathcal{Y}$ is* locally safe *if for every geometric point y of $\mathcal{Y}$ the neutral connected component of its automorphism group,* $\text{Aut}(y)$*, is unipotent.*

(b) *A morphism of algebraic stacks is* locally safe *if all its geometric fibers are.*

(c) *An algebraic stack (resp. a morphism of algebraic stacks) is* safe *if it is quasi-compact and locally safe.*

Remark 10.2.3. A safe algebraic stack is clearly QCA in the sense of Definition 1.1.8.

Theorem 10.2.4. *Let $\pi : \mathcal{Y} \to \mathcal{Y}'$ be a quasi-compact morphism of algebraic stacks. Then the functor $\pi_{\text{dR},*}$ is continuous if and only if π is safe. In the latter case $\pi_{\text{dR},*}$ strongly satisfies the projection formula.* [27]

This theorem is proved in Sect. 10.3 below.

Corollary 10.2.5. *If π is safe, the canonical map*

$$\pi_{\blacktriangle} \to \pi_{\text{dR},*}$$

is an isomorphism.

Proof. Both functors are continuous, and the map in question is an isomorphism on compact objects by Proposition 9.3.7. □

Corollary 10.2.6. *Let $\mathcal{Y}$ be a quasi-compact stack. Then the functor $\Gamma_{\text{dR}}(\mathcal{Y}, -)$ is continuous if and only if $\mathcal{Y}$ is safe.*

Corollary 10.2.7. *The following properties of a quasi-compact algebraic stack $\mathcal{Y}$ are equivalent:*

[27] See Sect. 7.7 for the explanation of what this means.

(i) $\text{D-mod}(\mathcal{Y})^c = \text{D-mod}_{\text{coh}}(\mathcal{Y})$;
(ii) $k_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y})^c$;
(iii) *The functor* $\Gamma_{\text{dR}}(\mathcal{Y}, -)$ *is continuous;*
(iv) *All objects of* D-mod$(\mathcal{Y})$ *are safe.*
(v) $\mathcal{Y}$ *is safe.*

Proof. By Corollary 10.2.6, (iii)$\Leftrightarrow$(v). Since $\mathcal{M}aps(k_{\mathcal{Y}}, -) = \Gamma_{\text{dR}}(\mathcal{Y}, -)$ we have (ii)$\Leftrightarrow$(iii). Clearly (i)$\Rightarrow$(ii). The equivalence (iii)$\Leftrightarrow$(iv) is tautological. It remains to prove that (iii)$\Rightarrow$(i).

The problem is to show that any $\mathcal{M} \in \text{D-mod}_{\text{coh}}(\mathcal{Y})$ is compact, i.e., the functor $\mathcal{M}aps(\mathcal{M}, -)$ is continuous. This follows from (iii) and the formula

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{Y})}(\mathcal{M}, \mathcal{M}') \simeq \Gamma_{\text{dR}}(\mathcal{Y}, \mathbb{D}^{\text{Verdier}}_{\mathcal{Y}}(\mathcal{M}) \overset{!}{\otimes} \mathcal{M}'), \qquad \mathcal{M} \in \text{D-mod}_{\text{coh}}(\mathcal{Y}),\ \mathcal{M}' \in \text{D-mod}(\mathcal{Y}),$$

which is the content of Lemma 7.3.5. □

10.2.8. *Characterization of safe objects of* D-mod$(\mathcal{Y})$. Let now $\mathcal{Y}$ be a QCA algebraic stack (in particular, it is quasi-compact).

Theorem 10.2.9. *Let $\mathcal{Y}$ be a QCA algebraic stack and $\mathcal{M} \in \text{D-mod}(\mathcal{Y})^b$. Then the following conditions are equivalent:*

(1) $\mathcal{M}$ *is safe;*
(2) *For any schematic quasi-compact morphism $\pi : \mathcal{Y}' \to \mathcal{Y}$ and any morphism $\varphi : \mathcal{Y}' \to S$ with S being a quasi-compact scheme, the object $\varphi_{\text{dR},*}\left(\pi^!(\mathcal{M})\right) \in \text{D-mod}(S)$ belongs to* $\text{D-mod}(S)^b$;
(3) *For any schematic quasi-compact morphism $\pi : \mathcal{Y}' \to \mathcal{Y}$ and any morphism $\varphi : \mathcal{Y}' \to S$ with S being a quasi-compact scheme, the object $\varphi_{\blacktriangle}\left(\pi^!(\mathcal{F})\right) \in \text{D-mod}(S)$ belongs to* $\text{D-mod}(S)^b$;
(4) *For any schematic quasi-compact morphism $\pi : \mathcal{Y}' \to \mathcal{Y}$ and any morphism $\varphi : \mathcal{Y}' \to S$ with S being a quasi-compact scheme, the canonical morphism*

$$\varphi_{\blacktriangle}\left(\pi^!(\mathcal{M})\right) \to \varphi_{\text{dR},*}\left(\pi^!(\mathcal{M})\right)$$

is an isomorphism;
(2′) - (4′) : *same as in* (2)-(4), *but π is required to be a finite étale map onto a locally closed substack of $\mathcal{Y}$.*
(5) $\mathcal{M}$ *belongs to the smallest (non-cocomplete) DG subcategory $\mathcal{T}(\mathcal{Y}) \subset \text{D-mod}(\mathcal{Y})$ containing all objects of the form $f_{\text{dR},*}(\mathcal{N})$, where $f : S \to \mathcal{Y}$ is a morphism with S being a quasi-compact scheme and $\mathcal{N} \in \text{D-mod}(S)^b$.*

Remark 10.2.10. Note, however, that the subcategory of safe objects in D-mod$(\mathcal{Y})$ is *not* preserved by the truncation functors.

10.3. **Proof of Theorem 10.2.4.**

10.3.1. *If $\pi_{\text{dR},*}$ is continuous then π is safe.* Up to passing to a field extension, we have to show that for any point $\xi : \text{pt} \to \mathcal{Y}'$ and any k-point y of the fiber $\mathcal{Y}_\xi$, the group $G := \text{Aut}(y)$ cannot contain a connected non-unipotent[28] algebraic subgroup $H \subset G$. We have a commutative

[28]If G were assumed affine, then "non-unipotent" could be replaced by "isomorphic to $\mathbb{G}_m$". Accordingly, at the end of Sect. 10.3.1 it would suffice to refer to the example of Sect. 7.1.4 instead of the example of Sect. 7.2.

diagram

$$\begin{tikzcd} BH \arrow[r, "f"] \arrow[d, "p"'] & \mathcal{Y} \arrow[d, "\pi"] \\ \mathrm{pt} \arrow[r, "\xi"] & \mathcal{Y}' \end{tikzcd}$$

in which f is the composition $BH \to BG \hookrightarrow \mathcal{Y}_\xi \to \mathcal{Y}'$. By assumption, $\pi_{\mathrm{dR},*}$ is continuous. By Sect. 6.1.7, $f_{\mathrm{dR},*}$ is also continuous since f is schematic and quasi-compact. So the composition $\pi_{\mathrm{dR},*} \circ f_{\mathrm{dR},*} = \xi_{\mathrm{dR},*} \circ p_{\mathrm{dR},*}$ is continuous. But $\xi_{\mathrm{dR},*}$ is continuous (by Sect. 6.1.7) and conservative (e.g., compute $\xi^! \circ \xi_{\mathrm{dR},*}$ by base change). Therefore $p_{\mathrm{dR},*}$ is continuous. This contradicts the Example of Sect. 7.2.

□

To prove the other statements from Theorem 10.2.4, we need to introduce some definitions.

10.3.2. *Unipotent group-schemes.* Let $\mathcal{X}$ be a prestack. A group-scheme over $\mathcal{X}$ is a group-like object $\mathcal{G} \in \mathrm{PreStk}_{/\mathcal{X}}$, such that the structure morphism $\mathcal{G} \to \mathcal{X}$ is schematic.

We shall say that $\mathcal{G}$ is unipotent if its pullback to any scheme gives a unipotent group-scheme over that scheme (a group-scheme is said to be unipotent if its geometric fibers are unipotent).

If $\mathcal{G}$ is smooth and unipotent, then the exponential map defines an isomorphism between $\mathcal{G}$ and the vector group-scheme of the corresponding sheaf of Lie algebras, as objects of $\mathrm{PreStk}_{/\mathcal{X}}$. This fact is stated in [Ra, Sect. XV.3 (iii)] without a proof, although the proof is not difficult. [29]

Lemma 10.3.3. *If $\mathcal{G}$ is a smooth unipotent group-scheme over $\mathcal{X}$, then the !-pullback functor* $\mathrm{D\text{-}mod}(\mathcal{X}) \to \mathrm{D\text{-}mod}(\mathcal{G})$ *is fully faithful.*

Proof. By the definition of D-mod on prestacks, it is sufficient to prove this fact when $\mathcal{X} = S$ is an affine DG scheme. Further Zariski localization reduces us to the fact that the pullback functor

$$\mathrm{D\text{-}mod}(S) \to \mathrm{D\text{-}mod}(S \times \mathbb{A}^n)$$

is fully faithful.[30]

□

10.3.4. *Unipotent gerbes.*

Definition 10.3.5. *We say that a morphism of prestacks $\mathcal{Z} \to \mathcal{X}$ is a* unipotent gerbe *if there exists an fppf cover $\mathcal{X}' \to \mathcal{X}$ such that $\mathcal{Z}' := \mathcal{Z} \underset{\mathcal{X}}{\times} \mathcal{X}'$ is isomorphic to the classifying stack of a smooth unipotent group-scheme over $\mathcal{X}'$.*

Lemma 10.3.6. *Let $\pi : \mathcal{Z} \to \mathcal{X}$ be a unipotent gerbe. Then the functor*

$$\pi^! : \mathrm{D\text{-}mod}(\mathcal{X}) \to \mathrm{D\text{-}mod}(\mathcal{Z})$$

is an equivalence.

---

[29]For our purposes, it will suffice to know that this fact when $\mathcal{X}$ is a scheme, generically on $\mathcal{X}$, in which case it is obvious.

[30]The reader who is not willing to use the isomorphism given by the exponential map on all of S, can prove the lemma by subdividing S into strata.

Proof. The statement is local in the fppf topology on $\mathcal{X}$, so we can assume that $\mathcal{Z} = B\mathcal{G}$ for some smooth unipotent group-scheme $\mathcal{G}$ over $\mathcal{X}$. Then

$$\text{D-mod}(\mathcal{Z}) \simeq \text{Tot}\left(\text{D-mod}(\mathcal{Z}^\bullet/\mathcal{X})\right),$$

where $\mathcal{Z}^\bullet/\mathcal{X}$ is the Čech nerve of $\mathcal{Z} \to \mathcal{X}$.

Each of the $n+1$ face maps $\mathcal{Z}^n/\mathcal{X} \to \mathcal{Z}^0/\mathcal{X}$ identifies with the natural projection

$$p_n : \mathcal{G}^{\times n} \to \mathcal{X},$$

where $\mathcal{G}^{\times n} = \mathcal{G} \underset{\mathcal{X}}{\times} \ldots \underset{\mathcal{X}}{\times} \mathcal{G}$.

Since $\mathcal{G}$ is unipotent, by Lemma 10.3.3 the functor $p_n^! : \text{D-mod}(\mathcal{X}) \to \text{D-mod}(\mathcal{G}^{\times n})$ is fully faithful. I.e., $p_n^!$ identifies $\text{D-mod}(\mathcal{X})$ with a full subcategory $\mathcal{C}^n \subset \text{D-mod}(\mathcal{G}^{\times n})$. Therefore,

$$\text{D-mod}(\mathcal{Z}) \simeq \text{Tot}\left(\text{D-mod}(\mathcal{Z}^\bullet/\mathcal{X})\right) \simeq \text{Tot}\left(\mathcal{C}^\bullet\right) \simeq \text{D-mod}(\mathcal{X}).$$

□

Assume that in the situation of Lemma 10.3.6, $\mathcal{Z}$ and $\mathcal{X}$ were algebraic stacks. In this case the functor $\pi_{\text{dR},*} : \text{D-mod}(\mathcal{Z}) \to \text{D-mod}(\mathcal{X})$ is defined.

Corollary 10.3.7. *Suppose that $\mathcal{Z}$ and $\mathcal{X}$ are algebraic stacks, and π is equidimensional. Under these circumstances, the functor $\pi_{\text{dR},*}$ is the inverse of $\pi^!$, up to a cohomological shift.*

Proof. Since π is smooth, the functor π^*_{dR} is defined and is the left adjoint of $\pi_{\text{dR},*}$. The assertion now follows from the fact that π^*_{dR} is isomorphic to $\pi^!$, up to a cohomological shift, see Sect. 6.1.9. □

10.3.8. *Nice open substacks.* To proceed with the proof of Theorem 10.2.4 and also for Theorem 10.2.9, we need the following variant of Lemma 2.5.2.

Lemma 10.3.9. *Let $\mathcal{Y} \neq \emptyset$ be a reduced classical algebraic stack over a field of characteristic 0 such that the automorphism group of any geometric point of $\mathcal{Y}$ is affine. Then there exists a diagram*

$$\begin{array}{ccccc} & & \mathcal{Z} & \to & X \times BG \\ & & \downarrow & & \\ \mathcal{Y} & \supset & \overset{\circ}{\mathcal{Y}} & & \end{array} \tag{10.1}$$

in which

- $\overset{\circ}{\mathcal{Y}} \subset \mathcal{Y}$ *is a non-empty open substack;*
- *the morphism $\pi : \mathcal{Z} \to \overset{\circ}{\mathcal{Y}}$ is schematic, finite, surjective, and étale;*
- *X is a scheme;*
- *G is a connected reductive algebraic group over k;*
- *the morphism $\psi : \mathcal{Z} \to X \times BG$ is a unipotent gerbe (in the sense of Definition 10.3.5).*

Remark 10.3.10. If $\mathcal{Y}$ is safe then G clearly has to be trivial.

Proof. Let $\overset{\circ}{\mathcal{Y}}$ be an open among the locally closed substacks given by Lemma 2.5.2. Let $\overset{\circ}{\mathcal{Y}} \to X'$, $X \to X'$ and $\mathcal{G}$ be the corresponding data supplied by that lemma.

Since we are in characteristic 0, the group-scheme $\mathcal{G}$ is smooth over X by Cartier's theorem. After shrinking X' and X we can assume that $\mathcal{G}$ is affine over X. After further shrinking, we can assume that the group-scheme $\mathcal{G}$ admits a factorization

$$1 \to \mathcal{G}_{un} \to \mathcal{G} \to \mathcal{G}_{red} \to 1, \tag{10.2}$$

where $\mathcal{G}_{un}$ and $\mathcal{G}_{red}$ are smooth group-schemes with $\mathcal{G}_{un}$ being unipotent and $\mathcal{G}_{red}$ being reductive and locally constant. After replacing X by a suitable étale covering, $\mathcal{G}_{red}$ becomes constant, i.e., isomorphic to $X \times G$ for some reductive algebraic group over k.

Now set

$$\mathcal{Z} := \overset{\circ}{\mathcal{Y}} \underset{X'}{\times} X = B\mathcal{G}.$$

We have a morphism $\mathcal{Z} = B\mathcal{G} \to B\mathcal{G}_{red} = X \times BG$. Thus we get a diagram (10.1), which has the required properties except that G is not necessarily connected. Finally, replace G by its neutral connected component G° and replace $\mathcal{Z}$ by $\mathcal{Z} \underset{BG}{\times} BG^\circ$. □

10.3.11. *Proof of Theorem 10.2.4.* By definition, we may assume that $\mathcal{Y}' = S$ is an affine DG scheme. In this case $\mathcal{Y}$ is quasi-compact (because π is). So by Noetherian induction, we can assume that the theorem holds for the restriction of π to any closed substack $\imath : \mathcal{X} \hookrightarrow \mathcal{Y}$, $\mathcal{X} \neq \mathcal{Y}$. Take $\mathcal{X} := (\mathcal{Y} - \overset{\circ}{\mathcal{Y}})$, where $\overset{\circ}{\mathcal{Y}}$ is as in Lemma 10.3.9. Then the exact triangle

$$\imath_{\mathrm{dR},*}(\imath^!(\mathcal{M})) \to \mathcal{M} \to \jmath_{\mathrm{dR},*}(\jmath^!(\mathcal{M})), \qquad \mathcal{M} \in \text{D-mod}(\mathcal{Y}),\ \jmath : \overset{\circ}{\mathcal{Y}} \hookrightarrow \mathcal{Y}$$

shows that it suffices to prove the theorem for $\pi|_{\overset{\circ}{\mathcal{Y}}}$. [31]

The morphism $p : \mathcal{Z} \to \overset{\circ}{\mathcal{Y}}$ is schematic, finite, surjective, and etale, so the functor $p_{\mathrm{dR},*} \circ p^!$, which is isomorphic to $p_{\mathrm{dR},*} \circ p^*_{\mathrm{dR}}$, contains $\mathrm{Id}_{\text{D-mod}(\mathcal{Y})}$ as a direct summand. Therefore it suffices to prove the theorem for the composition

$$\mathcal{Z} \to \overset{\circ}{\mathcal{Y}} \hookrightarrow \mathcal{Y} \xrightarrow{\pi} S. \tag{10.3}$$

Using Remark 10.3.10 and the assumption that S is a scheme, we can decompose the morphism (10.3) as

$$\mathcal{Z} \xrightarrow{f} X \xrightarrow{g} S,$$

where f is the canonical map $\mathcal{Z} \to X$.

It remains to show that each of the functors $f_{\mathrm{dR},*}$ and $g_{\mathrm{dR},*}$ has the properties stated in the theorem. This is clear for g as it is a morphism between quasi-compact schemes (see Sect. 5.2). For f, this follows from Lemma 10.3.6.

□

10.4. **Proof of Theorem 10.2.9.**

10.4.1. *Stability of safety.*

Lemma 10.4.2. *Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}$ be a morphism of QCA stacks.*

(a) *If $\mathcal{M}_1 \in \text{D-mod}(\mathcal{Y}_1)$ is safe then so is $\pi_{\mathrm{dR},*}(\mathcal{M}_1) \in \text{D-mod}(\mathcal{Y})$.*
(b) *If π is safe and $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$ is safe then so is $\pi^!(\mathcal{M}) \in \text{D-mod}(\mathcal{Y}_1)$.*

Proof.
(a) We need to show that the functor

$$\mathcal{N} \mapsto \Gamma_{\mathrm{dR}}(\mathcal{Y}, \pi_{\mathrm{dR},*}(\mathcal{M}_1) \overset{!}{\otimes} \mathcal{N}), \quad \mathcal{N} \in \text{D-mod}(\mathcal{Y})$$

is continuous. However, by Corollary 9.3.10, the right-hand side is isomorphic to

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}, \pi_{\mathrm{dR},*}(\mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{N})),$$

[31] Note that this step relies in Propositions 7.5.7 and 7.5.4.

i.e., $\Gamma_{\mathrm{dR}}(\mathcal{Y}_1, \mathcal{M}_1 \overset{!}{\otimes} \pi^!(\mathcal{N}))$, and the latter is continuous since $\mathcal{M}_1$ is safe.

(b) The functor

$$\mathcal{N}_1 \mapsto \Gamma_{\mathrm{dR}}(\mathcal{Y}_1, \pi^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{N}_1), \quad \mathcal{N}_1 \in \text{D-mod}(\mathcal{Y}_1)$$

is continuous because the projection formula

$$\Gamma_{\mathrm{dR}}(\mathcal{Y}_1, \pi^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{N}_1) \simeq \Gamma_{\mathrm{dR}}(\mathcal{Y}, \mathcal{M} \overset{!}{\otimes} \pi_{\mathrm{dR},*}(\mathcal{N}_1)),$$

is valid by Lemma 9.3.5, since $\pi_{\mathrm{dR},*} \simeq \pi_{\blacktriangle}$. □

Corollary 10.4.3. *Let $\imath_j : \mathcal{Y}_j \hookrightarrow \mathcal{Y}$, $j = 1, \ldots, n$, be locally closed substacks such that $\mathcal{Y} = \underset{j}{\cup} \mathcal{Y}_j$. Then an object $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$ is safe if and only if $\imath_j^!(\mathcal{M})$ is safe for each j.*

Proof. It suffices to consider the case where $n = 2$, $\mathcal{Y}_1$ is a closed substack, and $\mathcal{Y}_2 = (\mathcal{Y} - \mathcal{Y}_1)$. The "only if" statement holds by Lemma 10.4.2(b). To prove the "if" statement, consider the exact triangle $(\imath_1)_{\mathrm{dR},*}(\imath_1^!(\mathcal{M})) \to \mathcal{M} \to (\imath_2)_{\mathrm{dR},*}(\imath_2^!(\mathcal{M}))$. By Lemma 10.4.2,

$$(\imath_1)_{\mathrm{dR},*}(\imath_1^!(\mathcal{M})) \text{ and } (\imath_2)_{\mathrm{dR},*}(\imath_2^!)(\mathcal{M})$$

are both safe, so $\mathcal{M}$ is safe. □

10.4.4. *The mapping telescope argument.*

Lemma 10.4.5. *Let $\mathcal{T}(\mathcal{Y}) \subset \text{D-mod}(\mathcal{Y})$ be as in condition (5) of Theorem 10.2.9. Then $\mathcal{T}(\mathcal{Y})$ is closed under direct summands.*

Proof. The subcategory $\mathcal{T}(\mathcal{Y})$ has the following property: if $\mathcal{M} \in \mathcal{T}(\mathcal{Y})$ then the infinite direct sum

$$\mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \oplus \ldots$$

also belongs to $\mathcal{T}(\mathcal{Y})$. Indeed, it suffices to check this if $\mathcal{M} = f_{\mathrm{dR},*}(\mathcal{N})$, where $f : S \to \mathcal{Y}$ is a morphism with S being a quasi-compact scheme and $\mathcal{N} \in \text{D-mod}(S)^b$.

Now suppose that $\mathcal{M} \in \mathcal{T}(\mathcal{Y})$ and $\mathcal{N}' \in \text{D-mod}(\mathcal{Y})$ is a direct summand of $\mathcal{M}$. Let $p : \mathcal{M} \to \mathcal{M}$ be the corresponding projector. The usual formula

$$\mathcal{M}' = \operatorname{colim}(\mathcal{M} \xrightarrow{p} \mathcal{M} \xrightarrow{p} \mathcal{M} \to \ldots) = \operatorname{Cone}(\mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \oplus \ldots \to \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \oplus \ldots)$$

shows that $\mathcal{M}' \in \mathcal{T}(\mathcal{Y})$. □

10.4.6. *The key proposition.* We shall deduce Theorem 10.2.9 from the following proposition.

Let X be a quasi-compact scheme and G a connected algebraic group. Consider the algebraic stack $X \times BG$. Let $\varphi : X \times BG \to X$ and $\sigma : X \to X \times BG$ be the natural morphsims.

Let $\mathcal{T}_X(X \times BG) \subset \text{D-mod}(X \times BG)$ denote the smallest (non-cocomplete) DG subcategory containing all objects of the form $\sigma_{\mathrm{dR},*}(\mathcal{N})$, $\mathcal{N} \in \text{D-mod}(X)^b$.

Proposition 10.4.7. *For an object $\mathcal{M} \in \text{D-mod}(X \times BG)^b$ the following conditions are equivalent:*

(i) *$\mathcal{M} \in \mathcal{T}_X(X \times BG)$;*
(ii) *$\varphi_{\mathrm{dR},*}(\mathcal{M}) \in \text{D-mod}(X)^b$;*
(iii) *$\varphi_{\blacktriangle}(\mathcal{M}) \in \text{D-mod}(X)^b$.*

10.4.8. *Proof of Theorem 10.2.9 modulo Proposition 10.4.7.*

It is clear that (2)⇒(2′), (3)⇒(3′), (4)⇒(4′).

The direct image functor preserves boundedness from below (see Lemma 7.6.8), while the renormalized direct image functor preserves boundedness from above (see Proposition 9.4.2). So condition (4) implies (2) and (3), while condition (4′) implies (2′) and (3′).

By Lemma 10.4.2(a), condition (5) implies (1). Condition (1) implies condition (4) by Lemma 10.4.2(b) combined with Corollary 9.2.10.

Thus it remains to prove that (2′)⇒(5) and (3′)⇒(5).

Let $\mathcal{M} \in \text{D-mod}(\mathcal{Y})$ satisfy either (2′) or (3′). By Noetherian induction and Lemma 10.4.2, it suffices to show that there exists a non-empty open substack $\overset{\circ}{\mathcal{Y}}$ of $\mathcal{Y}$, such that the restriction $\mathcal{M}|_{\overset{\circ}{\mathcal{Y}}}$ satisfies condition (5).

We take $\overset{\circ}{\mathcal{Y}}$ to be as in Lemma 10.3.9. Consider the following diagram, in which the square is Cartesian:

$$
\begin{tikzcd}
\mathcal{Z}' \arrow[r, "\psi'"] \arrow[d, "\sigma'"'] & X \arrow[d, "\sigma"] \\
\mathcal{Z} \arrow[r, "\psi"] \arrow[d, "\pi"'] & X \times BG \\
\overset{\circ}{\mathcal{Y}}. &
\end{tikzcd}
\tag{10.4}
$$

The map $\psi' : \mathcal{Z}' \to X$ is a unipotent gerbe. By further shrinking X, we can assume that it admits a section; denote this section by g.

Note that $\mathcal{M}$ is a direct summand of $\pi_{\mathrm{dR},*}(\pi^!(\mathcal{M}))$. Hence, by Lemma 10.4.5, it suffices to show the following:

Proposition 10.4.9. *The object* $\pi^!(\mathcal{M}) \in \text{D-mod}(\mathcal{Z})$ *belongs to the smallest (non-cocomplete) DG subcategory of* $\text{D-mod}(\mathcal{Z})$ *containing all objects of the form* $f_{\mathrm{dR},*}(\mathcal{N})$*, where* $f = \sigma' \circ g$ *and* $\mathcal{N} \in \text{D-mod}(X)^b$.

Proof. We apply conditions (2′) or (3′) with $\mathcal{Y}' = \mathcal{Z}$, and ϕ being the composition

$$\mathcal{Z} \xrightarrow{\psi} X \times BG \xrightarrow{\varphi} X.$$

Consider the object $\psi_{\mathrm{dR},*}(\pi^!(\mathcal{M}))$. It is bounded because $\pi^!(\mathcal{M})$ is bounded, and $\psi_{\mathrm{dR},*}$ is an equivalence, which is t-exact up to a cohomological shift (see Lemma 10.3.6). Moreover,

$$\psi_{\blacktriangle}(\pi^!(\mathcal{M})) \simeq \psi_{\mathrm{dR},*}(\pi^!(\mathcal{M}))$$

because ψ, being a unipotent gerbe, is safe.

Consider now the objects

$$\varphi_{\mathrm{dR},*}\left(\psi_{\mathrm{dR},*}(\pi^!(\mathcal{M}))\right) \simeq \phi_{\mathrm{dR},*}(\pi^!(\mathcal{M})) \text{ and } \varphi_{\blacktriangle}\left(\psi_{\blacktriangle}(\pi^!(\mathcal{M}))\right) \simeq \phi_{\blacktriangle}(\pi^!(\mathcal{M}))$$

(note that the first isomorphism uses Sect. 7.8.6 in any of the (i), (ii) or (iii) versions).

Condition (2′) (resp., (3′)) implies that the former (resp., latter) object is in $\text{D-mod}(X)^b$. Hence, by the implications (ii)⇒(i) (resp., (iii)⇒(i)) in 10.4.7, we obtain that $\psi_{\mathrm{dR},*}(\pi^!(\mathcal{M}))$ belongs to the subcategory $\mathcal{T}_X(X \times BG)$.

Consider the Cartesian square in (10.4). Since $\psi_{\mathrm{dR},*}$ is an equivalence (Lemma 10.3.6), we obtain that the object $\pi^!(\mathcal{M})$ belongs to the smallest (non-cocomplete) DG subcategory of $\text{D-mod}(\mathcal{Z})$ containing all objects of the form $\sigma'_{\mathrm{dR},*}(\mathcal{N}')$, $\mathcal{N}' \in \text{D-mod}(\mathcal{Z}')^b$.

Recall that g denotes a section of the map ψ'. By Lemma 10.3.6, $\psi'_{\mathrm{dR},*}$ is an equivalence, and $g_{\mathrm{dR},*}$ is its left inverse. Hence, $g_{\mathrm{dR},*}$ is an equivalence as well. So, every object $\mathcal{N}' \in \text{D-mod}(\mathcal{Z}')^b$ is of the form $g_{\mathrm{dR},*}(\mathcal{N})$ for $\mathcal{N} \in \text{D-mod}(X)^b$, which implies the required assertion.

□

This finishes the proof Theorem 10.2.9 modulo Proposition 10.4.7.

10.5. **Proof of Proposition 10.4.7.** We already know from Sect. 10.4.8 that (ii)⇐(i)⇒(iii).

10.5.1. As a preparation for the proof of the implication (ii)⇒(i), we observe:

Lemma 10.5.2. *If* $\mathcal{M} \in \text{D-mod}(X \times BG)^{\geq r}$ *then*

$$\mathrm{Cone}\,(\mathcal{M} \to \sigma_{\mathrm{dR},*}(\sigma^*_{\mathrm{dR}}(\mathcal{M}))) \in \text{D-mod}(X \times BG)^{\geq r+1}.$$

Proof. Use that the fibers of σ are connected (because G is assumed to be connected). □

Corollary 10.5.3. *Let* $\mathcal{M} \in \text{D-mod}(X \times BG)^b$. *Then for every* $m \in \mathbb{Z}$ *there exists an exact triangle*

$$\mathcal{M} \to \mathcal{E} \to \mathcal{M}' \to \mathcal{M}[1] \tag{10.5}$$

with $\mathcal{E} \in \mathcal{T}_X(X \times BG)$, $\mathcal{M}' \in \text{D-mod}(X \times BG)^{\geq m} \cap \text{D-mod}(X \times BG)^b$.

10.5.4. Let $\mathrm{cd}(X)$ denote the cohomological dimension of $\text{D-mod}(X)$. Since X is a quasi-compact scheme $\mathrm{cd}(X) < \infty$ (and in fact, $\mathrm{cd}(X) \leq 2 \cdot \dim X$). By definition,

$$\mathrm{Ext}^j(\mathcal{N}, \mathcal{L}) = 0 \;\text{ if }\; \mathcal{N} \in \text{D-mod}(X)^{\geq m},\, \mathcal{L} \in \text{D-mod}(X)^{\leq n},\, j > n - m + \mathrm{cd}(X). \tag{10.6}$$

Lemma 10.5.5. *Let* $\mathcal{M}$ *be an object of* $\text{D-mod}(X \times BG)$ *such that* $\varphi_{\mathrm{dR},*}(\mathcal{M}) \in \text{D-mod}(X)^{\leq n}$, *and let* $\mathcal{M}'$ *be a bounded object in* $\text{D-mod}(X \times BG)^{\geq m}$. *Then*

$$\mathrm{Ext}^i(\mathcal{M}', \mathcal{M}) = 0 \textit{ for } i > n - m + \mathrm{cd}(X) + d,$$

where $d := \dim G$.

Proof. We can assume that $\mathcal{M}'$ lives in a single degree $\geq m$. Then $\mathcal{M}' = \varphi^*_{\mathrm{dR}}(\mathcal{N})[d]$ for some $\mathcal{N} \in \text{D-mod}(X)^{\geq m}$. Applying (10.6) to $\mathcal{L} = \varphi_{\mathrm{dR},*}(\mathcal{M})$ we see that the group

$$\mathrm{Ext}^i(\mathcal{M}', \mathcal{M}) = \mathrm{Ext}^{i-d}(\varphi^*_{\mathrm{dR}}(\mathcal{N}), \mathcal{M}) \simeq \mathrm{Ext}^{i-d}(\mathcal{N}, \varphi_{\mathrm{dR},*}(\mathcal{M}))$$

is zero if $i - d > n - m + \mathrm{cd}(X)$. □

10.5.6. We are now ready to prove the implication (ii)⇒(i) in Proposition 10.4.7.

Suppose that $\varphi_{\mathrm{dR},*}(\mathcal{M}) \in \text{D-mod}(X)^{\leq n}$. Apply Corollary 10.5.3 for $m = n + \mathrm{cd}(X) + d$. In the corresponding exact triangle (10.5) the morphism $\mathcal{M}' \to \mathcal{M}[1]$ is homotopic to 0 by Lemma 10.5.5. So

$$\mathcal{M} \oplus \mathcal{M}' \simeq \mathcal{E} \in \mathcal{T}_X(X \times BG).$$

Now the next lemma implies that $\mathcal{M} \in \mathcal{T}_X(X \times BG)$.

Lemma 10.5.7. *The subcategory* $\mathcal{T}_X(X \times BG) \subset \text{D-mod}(X \times BG)$ *is closed under direct summands.*

Proof. The same argument as in the proof of Lemma 10.4.5. □

10.5.8. *Proof of the implication (iii)⇒(i).*

Lemma 10.5.9. *For any connected algebraic group the functors*

$$\sigma^! : \text{D-mod}(X \times BG) \to \text{D-mod}(X) \text{ and } \varphi^! : \text{D-mod}(X) \to \text{D-mod}(X \times BG)$$

have left adjoints $\sigma_! : \text{D-mod}(X) \to \text{D-mod}(X \times BG)$ *and* $\varphi_! : \text{D-mod}(X \times BG) \to \text{D-mod}(X)$. *Moreover,*

$$\varphi_! \simeq \varphi_\blacktriangle[2(\dim(G) - \delta)], \quad \sigma_! \simeq \sigma_{\mathrm{dR},*}[\delta - 2\dim(G)]\,,$$

where δ *is the degree of the highest cohomology group of* $\Gamma_{\mathrm{dR}}(G, k_G)$.

Proof. By Corollary 8.3.4,

$$\text{D-mod}(X \times BG) \simeq \text{D-mod}(X) \otimes \text{D-mod}(BG),$$

and all functors involved in the lemma are continuous. Hence, they each decompose as

$$\mathrm{Id}_{\text{D-mod}(X)} \otimes \text{Corresponding functor for } BG.$$

So, it is sufficient to consider the case when $X = \mathrm{pt}$. The assertion in the latter case essentially follows from Example 9.1.6:

The fact that $\sigma_! \simeq \sigma_{\mathrm{dR},*}[\delta - 2\dim(G)]$ is evident: it suffices to compute both sides on $k \in \mathrm{Vect} = \text{D-mod}(\mathrm{pt})$. To show that

$$\Gamma_{\mathrm{dR},!}(BG, -) := \varphi_!$$

exists and satisfies

$$\Gamma_{\mathrm{dR},!}(BG, -) \simeq \Gamma_{\text{ren-dR}}(BG, -)[2(\dim(G) - \delta)],$$

it suffices to show that $\Gamma_{\mathrm{dR},!}(BG, -)$ is defined on the compact generator $\sigma_!(k)$ of $\text{D-mod}(BG)$, and

$$\Gamma_{\mathrm{dR},!}(BG, \sigma_!(k)) \simeq \Gamma_{\text{ren-dR}}(BG, \sigma_!(k))[2(\dim(G) - \delta)],$$

as modules over $\mathcal{M}aps_{\text{D-mod}}(\sigma_!(k), \sigma_!(k))$.

However, $\Gamma_{\mathrm{dR},!}(BG, \sigma_!(k)) \simeq k$, and required isomorphism was established in Example 9.1.6:

$$\Gamma_{\text{ren-dR}}(BG, \sigma_!(k)) \simeq \Gamma_{\text{ren-dR}}(BG, \sigma_{\mathrm{dR},*}(k))[-2\dim(G) + \delta] \simeq k[-2\dim(G) + \delta].$$

□

Lemma 10.5.9 allows to prove the implication (iii)⇒(i) from Proposition 10.4.7 by mimicking the arguments from Sect. 10.5.1. For example, the role of Lemma 10.5.2 is played by the following

Lemma 10.5.10. *If* $\mathcal{M} \in \text{D-mod}(X \times BG)^{\leq r}$ *then*

$$\mathrm{Cone}\left(\sigma_!(\sigma^!(\mathcal{M})) \to \mathcal{M}\right)[-1] \in \text{D-mod}(X \times BG)^{\leq r-1}\,.$$

□

10.6. **Proper morphisms of stacks.**

10.6.1. Recall the definition of a *proper* (but not necessarily schematic) morphism between algebraic stacks; see [LM, Definition 7.11].

As a simple application of the theory developed above, in this subsection we will prove the following:

Proposition 10.6.2. *Let* $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ *be a proper map between algebraic stacks. Then the functor* $\pi_{\mathrm{dR},*} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2)$ *sends* $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y}_1) \to \text{D-mod}_{\mathrm{coh}}(\mathcal{Y}_2)$.

The rest of this subsection is devoted to the proof of the proposition.

10.6.3. *Step 1.* First, we recall that the definition of properness includes separatedness. This implies that the groups of automorhisms of points of the geometric fibers of π are finite. In particular, π is safe.

By Theorem 10.2.4, $\pi_{\mathrm{dR},*}$ satisfies base change. This allows to assume that $\mathcal{Y}_2$ is an affine DG scheme. In this case $\mathcal{Y}_1$ is a safe QCA stack, and by Corollary 10.2.7

$$\text{D-mod}_{\mathrm{coh}}(\mathcal{Y}_1) = \text{D-mod}(\mathcal{Y}_1)^c.$$

Hence, it is enough to show that $\pi_{\mathrm{dR},*}$ sends $\text{D-mod}(\mathcal{Y}_1)^c$ to $\text{D-mod}(\mathcal{Y}_2)^c$.

The category $\text{D-mod}(\mathcal{Y}_1)^c$ is Karoubi-generated by the essential image of $\mathrm{Coh}(\mathcal{Y}_1)$ under the functor $\mathbf{ind}_{\text{D-mod}(\mathcal{Y}_1)}$. So, it is sufficient to show that the composition $\pi_{\mathrm{dR},*} \circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_1)}$ sends $\mathrm{Coh}(\mathcal{Y}_1)$ to $\text{D-mod}(\mathcal{Y}_2)^c$.

However, by Proposition 7.5.9,

$$\pi_{\mathrm{dR},*} \circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_1)} \simeq \mathbf{ind}_{\text{D-mod}(\mathcal{Y}_2)} \circ \pi_*^{\mathrm{IndCoh}}.$$

Hence, it is enough to show that the functor π_*^{IndCoh} sends $\mathrm{Coh}(\mathcal{Y}_1)$ to $\mathrm{Coh}(\mathcal{Y}_2)$.

10.6.4. *Step 2.* Consider the functor

$$\pi : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2).$$

We have a commutative diagram of functors

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{Y}_1)^+ \arrow[d, "\pi_*"'] & \mathrm{IndCoh}(\mathcal{Y}_1)^+ \arrow[l, "\Psi_{\mathcal{Y}_1}"'] \arrow[d, "\pi_*^{\mathrm{IndCoh}}"] \\ \mathrm{QCoh}(\mathcal{Y}_2)^+ & \mathrm{IndCoh}(\mathcal{Y}_2)^+, \arrow[l, "\Psi_{\mathcal{Y}_2}"'] \end{tikzcd}$$

where the horizontal arrows are equivalences.

Hence, it suffices to show that π_* sends $\mathrm{Coh}(\mathcal{Y}_1) \subset \mathrm{QCoh}(\mathcal{Y}_1)^+$ to $\mathrm{Coh}(\mathcal{Y}_2) \subset \mathrm{QCoh}(\mathcal{Y}_2)^+$.

By Corollary 1.4.5, π_* sends $\mathrm{QCoh}(\mathcal{Y}_1)^b$ to $\mathrm{QCoh}(\mathcal{Y}_2)^b$. Hence, it remains to show that π_* sends objects from $\mathrm{QCoh}(\mathcal{Y}_1)^\heartsuit \cap \mathrm{Coh}(\mathcal{Y}_1)$ to objects in $\mathrm{QCoh}(\mathcal{Y}_2)$ with coherent cohomologies.

However, the latter is the content of [F, Theorem 1] (see also [LM, Theorem 15.6(iv)], combined with [Ol, Theorem 1.2]).

11. More general algebraic stacks

11.1. **Algebraic spaces and LM-algebraic stacks.**

11.1.1. We define the notion of algebraic space as in [GL:Stacks], Sect. 4.1.1. We shall always impose the condition that our algebraic spaces be quasi-separated (i.e., the diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact). [32]

Thus, our definition is equivalent (the DG version) of that of [LM] (this relies on the DG version of Artin's theorem about the existence of an étale atlas, see Corollary 8.1.1 of [LM]).

An algebraic space is an algebraic stack in the sense of the definition of Sect. 1.1.1. Vice versa, an algebraic stack $\mathcal{X}$ is an algebraic space if and only if the following equivalent conditions hold:

- The underlying classical stack ${}^{cl}\mathcal{X}$ is a sheaf of sets (rather than groupoids).

[32] Note that the diagonal morphism of an algebraic space is always separated. In fact, for any presheaf of *sets* $\mathcal{X}$, the diagonal of the diagonal is an isomorphism.

- The diagonal map $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ induces a *monomorphism* at the level of underlying classical prestacks.

11.1.2. Let us recall that a morphism between prestacks $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ is called representable, if its base change by any affine DG scheme yields an algebraic space.

11.1.3. *LM-algebraic stacks.* We shall now enlarge the class of algebraic stacks as follows. We say that it is *LM-algebraic* if

- The diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable, quasi-separated, and quasi-compact.
- There exists a DG scheme Z and a map $f : Z \to \mathcal{Y}$ (automatically representable, by the previous condition) such that f is smooth and surjective.

11.1.4. *The extended QCA condition.* The property of being QCA makes sense for LM-algebraic stacks. We shall call these objects QCA LM-algebraic stacks.

We can now enlarge the class of QCA morphisms between prestacks accordingly. We shall say that a morphism is LM-QCA if its base change by an affine DG scheme yields QCA LM-algebraic stack.

11.2. **Extending the results.**

11.2.1. The basic observation that we make is that a quasi-compact algebraic space is automatically QCA. In particular, we obtain that quasi-compact representable morphisms are QCA.

Note also (for the purposes of considering D-modules) that a quasi-compact algebraic space is safe in the sense of Definition 10.2.2. In particular, a quasi-compact representable morphism is safe.

11.2.2. Let us now recall where we used the assumption on algebraic stacks that the diagonal morphism
$$\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$$
should be schematic.

In all three contexts (QCoh, IndCoh and D-mod) we needed the following property. Let S be an affine (or, more generally, quasi-separated and quasi-compact) DG scheme equipped with a smooth map $g : S \to \mathcal{Y}$. We considered the naturally defined functors
$$g^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(S), \quad g^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(S) \text{ and}$$
$$g^*_{\mathrm{dR}} : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(S).$$
We needed these functors to admit *continuous* right adjoints
$$g_* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(\mathcal{Y}), \quad g^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(\mathcal{Y}) \text{ and}$$
$$g_{\mathrm{dR},*} : \text{D-mod}(S) \to \text{D-mod}(\mathcal{Y}),$$
respectively.

Now, this was indeed the case, because the map g is itself schematic, quasi-separated and quasi-compact.

11.2.3. Now, we claim that the same is true for LM-algebraic stacks. Indeed, if $\mathcal{Y}$ is an LM-algebraic stack and S is a DG scheme, then any morphism $g : S \to \mathcal{Y}$ is representable, quasi-separated and quasi-compact.

In particular, if S is an affine (or, more generally, quasi-separated and quasi-compact) DG scheme, the morphism g is QCA (and safe).

We obtain that Corollary 1.4.5 implies the corresponding fact for g_*.

Corollary 3.7.13, applied after a base change by all maps $f : Z \to \mathcal{Y}$ where $Z \in \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$, implies the required property of g_*^{IndCoh}.

Finally, Theorem 10.2.4, again applied after a base change by all maps $f : Z \to \mathcal{Y}$ where $Z \in \mathrm{DGSch}^{\mathrm{aff}}_{\mathrm{aft}}$, implies the required property of $g_{\mathrm{dR},*}$.

11.2.4. Another ingredient that went into the proofs of the main results was Proposition 2.3.4. However, it is easy to see that its proof works for LM-algebraic stacks with no modification.

The rest of the ingredients in the proofs are without change.

11.2.5. In application to the category $\mathrm{QCoh}(-)$, we have the following generalization of Theorem 1.4.2:

Theorem 11.2.6.

(a) *Suppose that an LM-algebraic stack $\mathcal{Y}$ is QCA. Then the functor $\Gamma : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}$ is continuous. Moreover, there exists an integer $n_{\mathcal{Y}}$ such that $H^i(\Gamma(\mathcal{Y}, \mathcal{F})) = 0$ for all $i > n_{\mathcal{Y}}$ for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$.*

(b) *Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a LM-QCA morphism between prestacks. Then the functor $\pi_* : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2)$ is continuous.*

11.2.7. In application to IndCoh, we have:

Theorem 11.2.8. *Suppose that an LM-algebraic stack $\mathcal{Y}$ is QCA. Then the category $\mathrm{IndCoh}(\mathcal{Y})$ is compactly generated, and its subcategory of compact objects identifies with $\mathrm{Coh}(\mathcal{Y})$.*

In particular, the statements of Corollary 4.2.3 and Theorem 4.3.1 hold for LM-algebraic stacks as well.

11.2.9. In application to D-modules, we have:

Theorem 11.2.10.

(a) *If an LM-algebraic stack $\mathcal{Y}$ is QCA then the category $\text{D-mod}(\mathcal{Y})$ is compactly generated. An object of $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ is compact if and only if it is safe.*

(b) *Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a quasi-compact morphism between LM-algebraic stacks. Then the functor $\pi_{\mathrm{dR},*}$ is continuous if and only if π is safe.*

Note that in Theorem 10.2.9(2)-(4) we can replace the words “schematic” by ”representable”.

References

[AG] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves, and the geometric Langlands conjecture*, arXiv:1201.6343.

[BeV] A. Beilinson and V. Vologodsky, *A DG guide to Voevodksy's motives*, Geom. Funct. Anal. **17** (2008), no. 6, 1709–1787.

[BFN] D. Ben-Zvi, J. Francis and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. **23** (2010), no. 4, 909–966.

[F] G. Faltings, *Finiteness of coherent cohomology for proper fppf stacks*, J. Algebraic Geom. **12** (2003), no. 2, 357–366.

[FG] J. Francis and D. Gaitsgory, *Chiral Koszul duality*, Selecta Mathematica (New Series), **18** (2012), no. 1, 27–87.

[GL:DG] Notes on Geometric Langlands, *Generalities on DG categories*, available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GL:Stacks] Notes on Geometric Langlands, *Stacks*, available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GL:QCoh] Notes on Geometric Langlands, *Quasi-coherent sheaves on stacks*, available at http://www.math.harvard.edu/ gaitsgde/GL/.

[IndCoh] D. Gaitsgory, *Ind-coherent sheaves*, arXiv:1105.4857.

[GR1] D. Gaitsgory and N. Rozenblyum, *Crystals and D-modules*, arXiv:1111.2087.

[GR2] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry*, in preparation.

[LM] G. Laumon, L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3 Folge, A Series of Modern Surveys in Mathematics), **39**, Springer-Verlag, Berlin, 2000.

[Lu1] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, **170**, Princeton University Press, Princeton, NJ, 2009.

[Lu2] J. Lurie, *Higher Algebra*, available at http://www.math.harvard.edu/˜lurie/.

[Lu3] J. Lurie, *DAG-VII: Spectral Schemes*, available at http://www.math.harvard.edu/˜lurie/.

[N] A. Neeman, *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 5, 547–566.

[Ne] A. Neeman, *Non-left-complete derived categories*, Math. Research Letters, **18** (2011) no. 5, 827–832.

[Ol] M. Olsson, *On proper coverings of Artin stacks*, Adv. Math. **198** (2005), no. 1, 93–106.

[Ra] M. Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes,* Lecture Notes in Mathematics, **119** Springer-Verlag, Berlin-New York 1970.

[SGA2] Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux, Augmenté d'un exposé de Michèle Raynaud. Documents Mathématiques **4**, Société Mathématique de France, Paris, 2005. arXiv:math/0511279.

[TT] R. Thomason and T. Trobaugh, *Higher algebraic K -theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, 247435, Progr. Math., **88** (1990).